

# Spherical Tiling by 12 Congruent Pentagons

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## Abstract

The tiling of the 2-dimensional sphere by congruent triangles have been extensively studied, and the edge-to-edge tilings have been completely classified. However, not much is known of the tiling by other congruent polygons. In this paper, we classify the simplest case, which is the edge-to-edge tiling of the 2-dimensional sphere by 12 congruent pentagons. We find one major class allowing two independent continuous parameters and four classes of isolated examples. The isolated examples are reduced to the existence of some spherical pentagons with specific edge length and angle relations. This is done by first separately classifying the combinatorial, edge length, and angle aspects of the tiling, and then combining the respective classifications together.

## 1 Introduction

Tilings have been studied by mathematicians for more than 100 years. Some major achievements include the solution to Hilbert's 18th problem [5, 12, 15], the classification of wallpaper groups [9, 13] and crystallographic groups [4, 8, 16], the classification of isohedral tilings of the plane [10], and the classification of edge-to-edge monohedral tilings of the 2-sphere by triangles [6, 19, 22]. We refer the reader for a more complete history of the subject to the excellent 1987 monograph by Grünbaum and Shephard [11].

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In this paper, we will study *monohedral* tilings, which means that all the tiles are geometrically congruent. For monohedral tilings of the plane by polygon, it is easy to see that any triangle or quadrilateral can be the tile. It is also known that convex tiles cannot have more than six edges, there are exactly three classes of convex hexagon tiles [14] and at least 14 classes of convex pentagon tiles [11, 18]. Of course there are many classes of non-convex tiles, including many with more than six edges. However, most works on plane tiling assume edges are straight. Although our main result assumes straight line edges (actually great arcs), it does not assume convexity, and most of the results in this paper allow any reasonably nice curve to be the edges. In principal, it is possible to extend the main result to the cases the edges are not straight, where the congruence means the same edge lengths, the same angles and the same adjacency relations, but not the identification under an isometry.

The tilings we study are also *edge-to-edge*, which means that no vertex of a tile lies in the interior of an edge of another tile. The assumption is needed to simplify the classification of tiling patterns, or all the possible ways the tiles fit together to cover the whole plane (actually the sphere in our case). It is easy to see that any triangle and quadrilateral can be the tile of an edge-to-edge monohedral tiling of the plane. Moreover, 8 of the 14 convex pentagon classes can be used for edge-to-edge tilings [3]. On the other hand, there is no general classification of plane tiling patterns, even for edge-to-edge and monohedral tilings by convex polygons. The known classifications often assume certain symmetry [10], or certain special geometric property of the tiles [20, 21]. The difficulty with classifying the plane tilings is that the number of tiles is infinite. By restricting to the spherical tiling, the problem becomes finite and more manageable. In fact, our classification is only for the minimum spherical pentagon tiling. On the other hand, our classification does not assume any symmetry or special geometric property.

We also restrict the study to the tilings such that all vertices have *degree*  $\geq 3$ . This avoids some trivial examples obtained by artificially adding extra vertices to the edges, or getting new tilings by “wiggling modifications” of the edges. The assumption provides another simplification of the classification.

Now we come to the subject of this paper, the edge-to-edge monohedral tilings of the sphere by polygons (subject to degree  $\geq 3$  vertex condition). It is not hard to see that the tile can only be triangle, quadrilateral, or pentagon [23, Proposition 4]. The first work in this direction is Sommerville’s 1923 partial classification of edge-to-edge monohedral spherical tilings by triangles

[19]. Davies [6] outlined a complete classification in 1967, and the complete proof was finally given by Ueno and Agaoka [22] in 2002. See [7] for the further study of monohedral non-edge-to-edge spherical tilings by triangles. As for quadrilateral spherical tilings, Ueno and Agaoka [23] showed that the classification can be very complicated. For works on the spherical tilings by special types of quadrilaterals, see [1, 2].

As far as we know, there have been virtually no study of tilings of the sphere by congruent pentagons. However, we are of the opinion that the spherical pentagon tilings should be easier to study than the quadrilateral ones, because among triangle, quadrilateral and pentagon, pentagon is the “other extreme”. We feel that the spherical pentagon should be compared to the planar hexagon, and the spherical quadrilateral should be compared the planar pentagon. To test our conviction, we started by classifying the minimal case of the edge-to-edge monohedral tiling of the sphere by pentagons. The fruit of this investigation is the main result of this paper.

**Main Theorem** *Any edge-to-edge spherical tiling by 12 congruent pentagons must belong to one of the five classes in Figure 1, subject to the condition that the sum of angles at any vertex is  $2\pi$ .*

Here is how to read the tiles in Figure 1: Combinatorially, the tilings are the dodecahedron, and are illustrated on the flat plane after punching a hole in one tile. The center tile shows the angles in a tile, with the  $\delta$  angle highlighted by a circle. The circles in the other tiles denote the  $\delta$  angle. Once the  $\delta$  angle is located in a tile, the other angles are determined by the angle arrangement orientation. The white tiles have the same orientation as the center tile, and the shaded tiles have opposite orientation as the center tile. For example, the angles in all tiles in  $T_1$  and  $T_5$  are orientated in the same way, and the angles in the punched tile in  $T_4$  have different orientation from the center tile. As for the edge lengths, the usual solid lines have the same edge length  $a$ , the thick lines have the same length  $b$ , and the dotted lines have the same length  $c$ .

Although the classification contains five classes, only the fifth class is the essential one, because the other four classes only provide isolated examples (and perhaps the regular dodecahedron only). Figure 2 is this essential class in its full glory.

The tile in the “essential class”  $T_5$  is illustrated on the left of Figure 3.

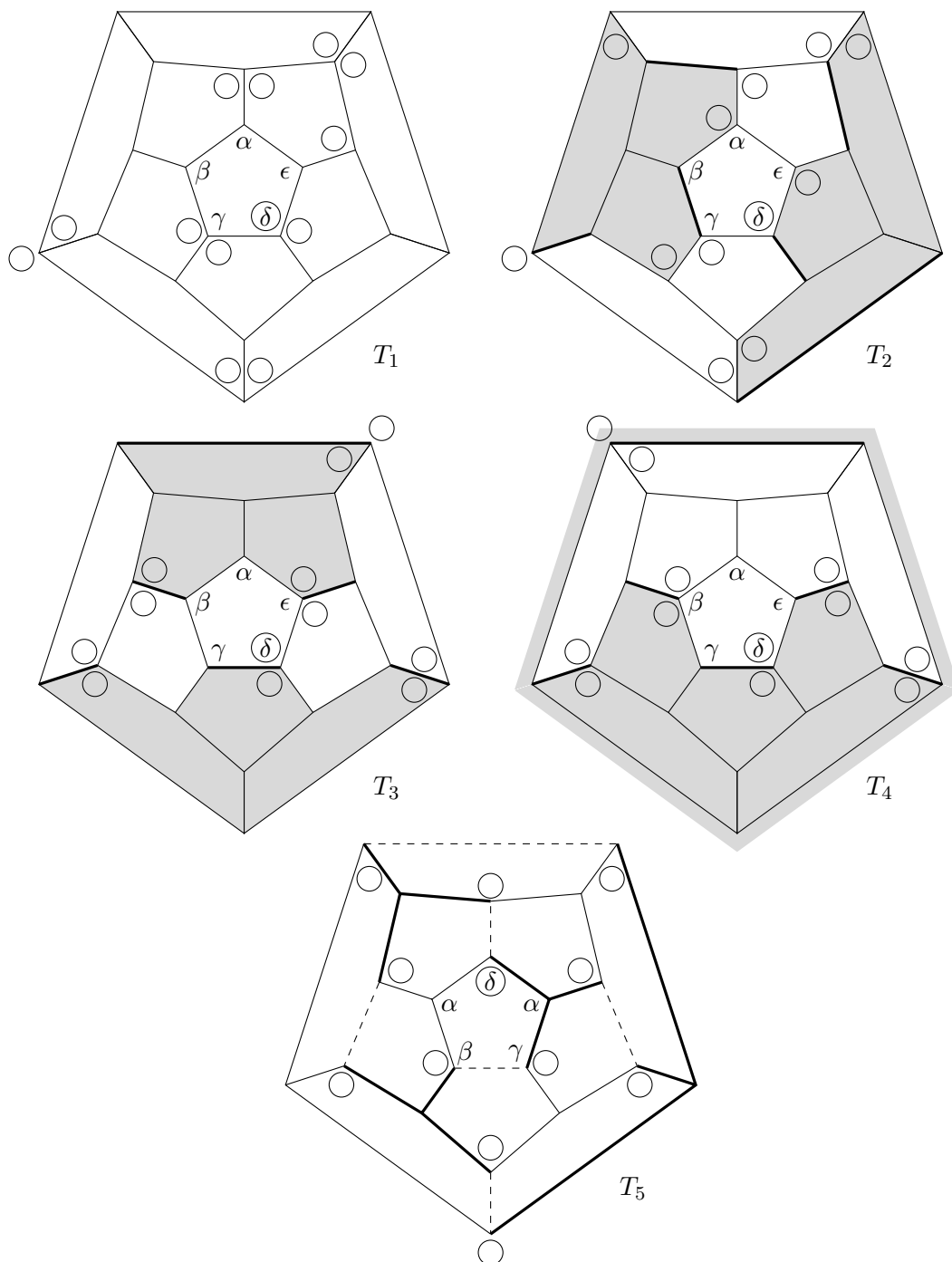


Figure 1: All the possible spherical tilings by 12 congruent pentagons.

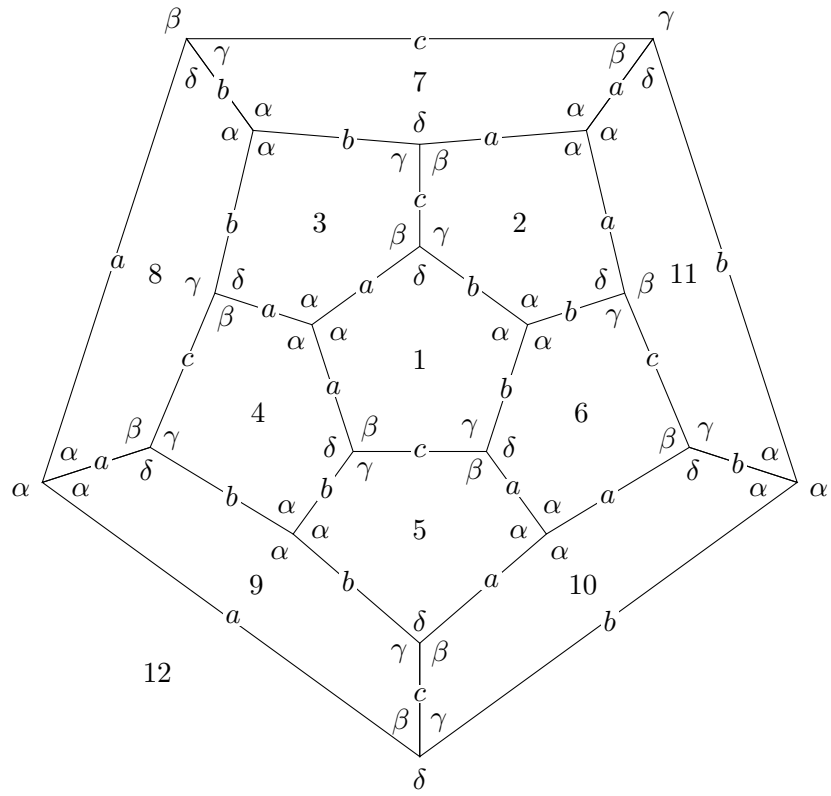


Figure 2: The general class  $T_5$  of spherical tilings by 12 congruent pentagons.

The angle sum condition is

$$3\alpha = \beta + \gamma + \delta = 2\pi.$$

The second equation is equivalent to the area of the pentagon being  $\frac{\pi}{3}$ .

It is easy to see that the tiles in  $T_5$  form a 2-parameter family. Divide the tile as in the middle of Figure 3. Since  $\alpha = \frac{2\pi}{3}$  is already fixed, the area  $A(a)$  of the triangle  $\triangle ABC$  is completely determined by  $a$ .  $A(a)$  is a strictly increasing function, and the area of  $\triangle ADE$  is  $A(b)$ . When  $A(a) + A(b) \leq \frac{\pi}{3}$  and  $A(a) + A(b)$  is sufficiently close to  $\frac{\pi}{3}$ , we can always find a suitable angle  $\angle CAD$  (equivalent to finding suitable  $\delta = \angle BAE$ ) so that the area of the pentagon is  $\frac{\pi}{3}$ . This shows that we can freely choose  $a$  and  $b$  within certain range to get a tile that fits into the tiling, and the tile is completely determined by the choice.

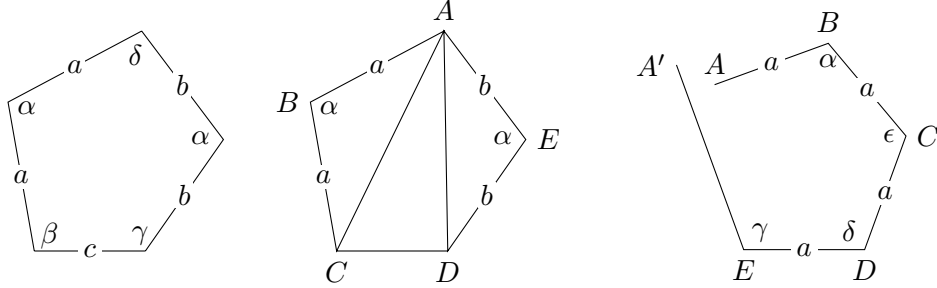


Figure 3: The pentagon tile.

Suppose we put the cube in Figure 4 inside a sphere, so that the cube and the sphere have the same center. If we put a light source at the center, then the projection of the cube to the sphere is an example of  $T_5$ -tiling, where  $\beta + \gamma = \delta = \pi$ .

For the tiling  $T_2$ , the angle sum condition is  $3\alpha = 2\beta + \epsilon = 2\gamma + \delta = \alpha + \delta + \epsilon = 2\pi$ . This means

$$\alpha = \frac{2\pi}{3}, \quad \beta = \pi - \frac{1}{2}\epsilon, \quad \gamma = \frac{\pi}{3} + \frac{1}{2}\epsilon, \quad \delta = \frac{4\pi}{3} - \epsilon,$$

and should lead to only isolated examples. The reason is illustrated on the right of Figure 3. Suppose  $a$  and  $\epsilon$  are given. Fix a point  $A$  on the sphere and a direction for the edge  $AB$ . Starting from  $A$  and travelling along the given direction by distance  $a$ , we arrive at  $B$ . Turning an angle  $\alpha$  at  $B$

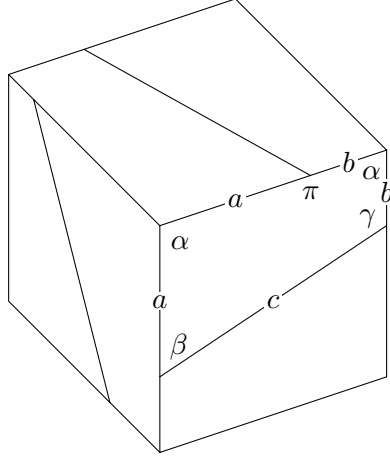


Figure 4: A spherical pentagon tiling.

and travelling by distance  $a$ , we arrive at  $C$ . Turning an angle  $\epsilon$  at  $C$  and travelling by distance  $a$ , we arrive at  $D$ . Turning an angle  $\delta = \frac{4\pi}{3} - \epsilon$  at  $D$  and travelling by distance  $a$ , we arrive at  $E$ . Turning an angle  $\gamma = \frac{\pi}{3} + \frac{1}{2}\epsilon$  at  $E$  and travelling along, we get a line (actually a great arc)  $EA'$ . Then we want  $A$  to lie on the line  $EA'$ , which imposes one relation between  $a$  and  $\epsilon$ , or reduce the number of free parameters to one. Once  $A$  lies on  $EA'$ , we further need the pentagon to have area  $\frac{\pi}{3}$ , which is the same as  $\angle EAB = \beta = \pi - \frac{1}{2}\epsilon$ . This imposes one more condition and we are left with no free parameter. Therefore the pentagon tiles fitting into the tiling  $T_2$  must appear in isolated way.

Similar discussion can be made for  $T_3$  and  $T_4$ . For  $T_1$ , we have even one less degree of freedom. The problem is crystalized below (the angles are renamed in four cases in order to get the same angle relations).

**Problem.** Find spherical pentagons in Figure 5 satisfying  $3\alpha = 2\beta + \gamma = 2\delta + \epsilon = \alpha + \gamma + \epsilon = 2\pi$ . Are there any examples besides the pentagon in the regular dodecahedron?

To prove the main theorem, we first separate and classify different aspects of the tiling. At the most crude level, we study tilings by combinatorially congruent tiles in Section 2, which means that all tiles have the same number of edges, and edge lengths and angles are ignored. Proposition 5 says that the dodecahedron is the unique spherical tiling by 12 combinatorially congruent pentagons. Combinatorial tilings have been extensively studied. See [17] for

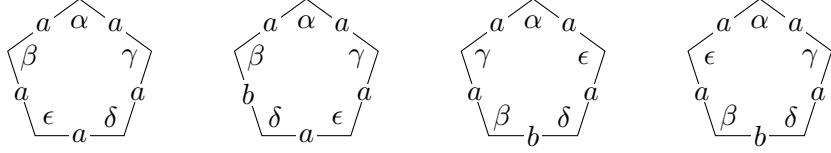


Figure 5: Must these spherical pentagons be regular?

a recent survey. Our study is actually a topological one, in the sense that the lines do not even need to be straight.

In Sections 3 and 4, we study tilings by edge congruent tiles, which means that all tiles have the same edge length combination and arrangement. Propositions 9, 10, 12, 11 (plus the case all edges have the same length) completely classify spherical tilings by 12 edge congruent pentagons.

In Sections 5 and 6, we study tilings by angle congruent tiles, which means that all tiles have the same angle combination and arrangement. Proposition 17 classifies the numerics in spherical tilings by 12 angle congruent pentagons. Propositions 18, 19, 20 then further classify the locations of the angles in three major cases. The complete classification would involve another major but rather complicated case. Since this case will not be needed in the proof of the main classification theorem, we only present some examples instead of the whole proof.

In Section 7, we combine the edge congruent and angle congruent classifications to prove the main theorem. While our edge congruent and angle congruent results do not require geometry, in that the edges are not necessarily great arcs, in combining the two, we use geometry to cut down the possible number of edge-angle combinations. So our main theorem does require the edges to be great arcs. If we drop the great arc requirement, it is possible to have more varieties of spherical pentagon tilings.

Finally, we would like to thank the referees for the careful reading of the paper and many helpful suggestions.

## 2 Combinatorial Conditions

In this section, we study the combinatorial aspect of tiling. This means that we ignore the edge length and the angle, and only require the tiles to have the same number of edges. Most results of the section are well known.

The results certainly apply to the edge-to-edge tilings as defined in [11].



However, some results can be applied to more general tilings. Specifically, we consider a connected graph nicely embedded in the sphere. We assume that all vertices in the graph have degree (or valence)  $\geq 3$ . The graph divides the sphere into regions, which become our tiles. In other words, our tilings always come from suitable graphs.

The interior of a tile is a connected component of the complement of the graph. A tile is the closure of its interior. The boundary of a tile is what is between the tile and its interior, and is a union of some edges in the graph.

The exclusion of vertices of degree 2 means that we do not view an  $n$ -gon as an  $(n+k)$ -gon by artificially choosing another  $k$  points on the boundary as the vertex. Therefore all tiles have naturally well-defined vertices and edges that are already given in the graph, and the tiling is naturally edge-to-edge.

Figure 6 shows some possible tiles. The first one does not have any boundary points identified. The others have some boundary points (even boundary intervals) identified. In case some boundary intervals are identified, the interior of the tile is different from the topological version of the interior.

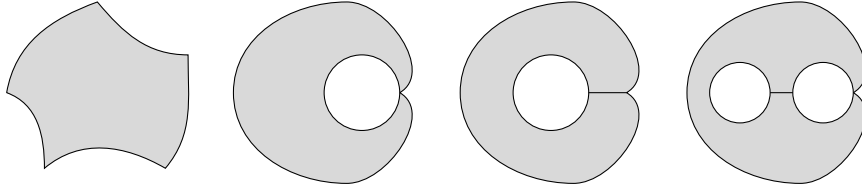


Figure 6: Tiles.

Figure 7 shows some examples that are not tiles. They are unions of several tiles at finitely many boundary points.

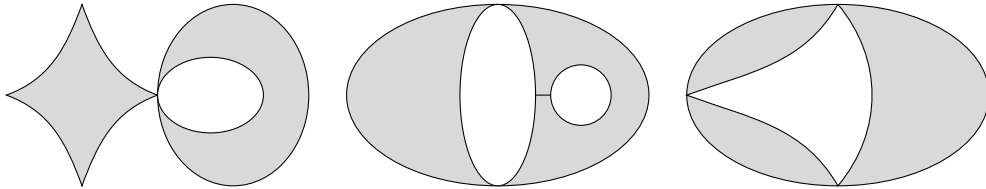


Figure 7: Not tiles.

**Lemma 1.** *In any (naturally generated) spherical tiling, there is at least one tile for which no boundary points are identified.*

*Proof.* We call a closed subset of the sphere a *simple region* if it is enclosed by a nicely embedded simple closed curve. The boundary points of a simple region are not identified. The lemma basically says that some tile is a simple region.

It is a topological fact that if  $R$  is the whole sphere or a simple region, and  $P$  is a tile in  $R$  with some boundary points identified, then the closure of some connected component of  $R - P$  is a simple region.

Let  $P_1$  be a tile, such that some boundary points are identified. Then the complement  $S^2 - P_1$  has a connected component  $C_1$ , such that the closure  $\bar{C}_1$  is a simple region, and is actually a union of tiles. If  $\bar{C}_1$  is a tile, then we are done. Otherwise,  $\bar{C}_1$  contains a tile  $P_2$  as a proper subset. If  $P_2$  is a simple region, then we are also done. If  $P_2$  is not a simple region, then some boundary points of  $P_2$  are identified, and the complement  $\bar{C}_1 - P_2$  has a connected component  $C_2$ , such that the closure  $\bar{C}_2$  is a simple region. Keep going. As long as we do not encounter tiles that are simple regions along the way, we find a strictly decreasing sequence of simple regions  $\bar{C}_1 \supset \bar{C}_2 \supset \bar{C}_3 \supset \dots$ . Since there are only finitely many tiles in the tiling, the sequence must stop. This means that some tile found in the process must be a simple region.  $\square$

An immediate consequence of the lemma is that in case all tiles are congruent (full congruence, including edge length and angle), the boundary points of tiles are never identified. Since the property is needed in the subsequent discussion, we include it in the following definition.

**Definition 2.** A tiling is *combinatorially monohedral* if all tiles have the same number of edges and every tile has no boundary points identified. The tiles in such a tiling are *combinatorially congruent*.

Let  $f$ ,  $e$ ,  $v$  be the numbers of tiles (or faces), edges and vertices in a combinatorially monohedral tiling. Let  $n$  be the number of edges in a tile. Then we have the Euler equation and the Dehn-Sommerville equation

$$v - e + f = 2, \quad 2e = nf.$$

Note that the Dehn-Sommerville equation assumes that each edge is shared by two *different* faces. This follows from the condition that no boundary points of a tile are identified. (Actually we only need that no boundary

intervals are identified.) Moreover, if  $v_i$  is the number of vertices of degree  $i$ , then

$$v = v_3 + v_4 + v_5 + \cdots, \quad 2e = 3v_3 + 4v_4 + 5v_5 + \cdots.$$

It is a simple consequence of all these equations that  $n = 3, 4, 5$ . See Proposition 4 of [23].

Now we concentrate on the case  $n = 5$ . The equations become

$$2e = 5f, \quad 2v = 3f + 4, \quad v_3 - 20 = 2v_4 + 5v_5 + 8v_6 + \cdots. \quad (2.1)$$

**Lemma 3.** *Any spherical tiling by combinatorially congruent pentagons contains a tile in which at least four vertices have degree 3.*

*Proof.* Let  $t_j$  be the number of degree 3 vertices in the  $j$ -th tile. Then  $3v_3 = \sum t_j$ . On the other hand, we have

$$\begin{aligned} 3f &= 2v - 4 = 2v_3 + 2v_4 + 2v_5 + 2v_6 + \cdots - 4 \\ &\leq 2v_3 + 2v_4 + 5v_5 + 8v_6 + \cdots - 4 \\ &= 2v_3 + (v_3 - 20) - 4 < 3v_3 = \sum t_j. \end{aligned}$$

This implies that some  $t_i > 3$ . □

**Lemma 4.** *A spherical tiling by combinatorially congruent pentagons has at least 12 tiles. Moreover, the minimum 12 is reached if and only if all vertices have degree 3.*

*Proof.* The third equation in (2.1) implies that  $v \geq v_3 \geq 20$ , and  $v = 20$  if and only if all vertices have degree 3. On the other hand, by the second equation,  $v \geq 20$  is the same as  $f \geq 12$ , and  $v = 20$  is the same as  $f = 12$ . □

**Proposition 5.** *The spherical tiling by 12 combinatorially congruent pentagons is uniquely given by Figure 8.*

*Proof.* By Lemma 4, all vertices have degree 3.

We start with a tile  $P_1$ , which has five edges. Each of the five edges is shared by  $P_1$  and exactly another tile. We denote these tiles by  $P_2, P_3, P_4, P_5, P_6$  and assume they are arranged in counterclockwise order.

Consider the vertex  $A$  of  $P_1$  on the left of Figure 9. It is also a vertex of  $P_2$  and  $P_3$ . Since  $A$  has degree 3, besides the two edges of  $P_1$  connected to  $A$ , there is exactly one more edge  $x$  also connected to  $A$ . The three edges

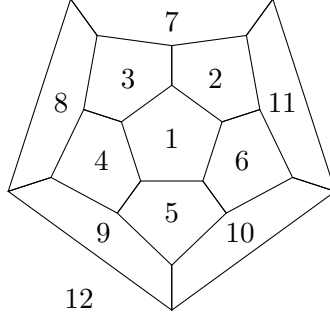


Figure 8: Combinatorial structure of the minimal pentagon tiling.

divide the neighborhood of  $A$  into three corners, each belonging to one tile. This local picture implies that  $x$  is a common edge of  $P_2$  and  $P_3$ . The similar situation happens between  $P_3$  and  $P_4$ , between  $P_4$  and  $P_5$ , between  $P_5$  and  $P_6$ , and between  $P_6$  and  $P_2$ . Therefore the first six tiles are glued together in the way described in Figure 8. Of course this does not yet exclude the possibility that additional identification may happen among these tiles. But at least the relations described in Figure 8 already exist between these tiles.

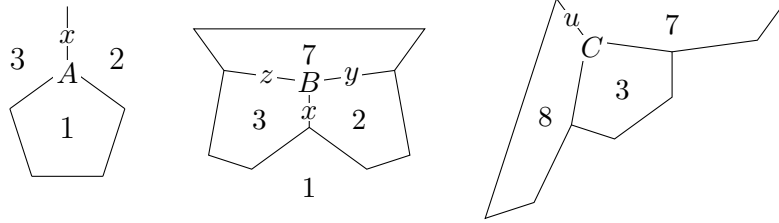


Figure 9: Neighborhood of degree 3 vertex.

The edge  $x$  has another end vertex  $B$  besides  $A$ . The three edges connected to  $B$  are  $x$ ,  $y$ ,  $z$ , where  $y$  is the edge of  $P_2$  adjacent to  $x$  and  $z$  is the edge of  $P_3$  adjacent to  $x$ . See middle of Figure 9. Now  $y$  is shared by  $P_2$  and another tile  $P_7$ , and  $z$  is shared by  $P_3$  and another tile  $P'_7$ . The three edges  $x$ ,  $y$ ,  $z$  divide the neighborhood of  $B$  into three corners, each belonging to one tile. Since two corners already belong to  $P_2$  and  $P_3$ , the third corner belongs to the same tile. This implies that  $P_7 = P'_7$ . Thus the 7-th tile is established and is related to  $P_2$  and  $P_3$  as in Figure 8. Similarly,  $P_8$  can be defined from  $P_3$  and  $P_4$ ,  $P_9$  can be defined from  $P_4$  and  $P_5$ ,  $P_{10}$  can be defined from  $P_5$  and  $P_6$ , and  $P_{11}$  can be defined from  $P_6$  and  $P_2$ .

The edge  $z$  has another end vertex  $C$  besides  $B$ . It is also a vertex of  $P_7$  and  $P_8$ . There are three edges connected to  $C$ . Two are edges of  $P_3$  and the third one is  $u$  on the right of Figure 9. The situation is the same as the left of Figure 9, with  $P_3, P_7, P_8, C, u$  in place of  $P_1, P_2, P_3, A, x$ . We get the same conclusion that the edge  $u$  is shared by  $P_7$  and  $P_8$ . We can deduce similar relations between  $P_8$  and  $P_9$ , etc. Then the tiles  $P_1$  through  $P_{11}$  are glued together as in Figure 8.

After constructing the first eleven tiles, we find that  $P_7, P_8, P_9, P_{10}, P_{11}$  have one “free” edge each. The free edge of  $P_7$  is shared by  $P_7$  and another tile  $P_{12}$ . The free edge of  $P_8$  is shared by  $P_8$  and another tile  $P'_{12}$ . The situation is similar to the relation between  $P_2, P_3, P_7, P'_7$  as described in the middle of Figure 9. We get the same conclusion that  $P_{12} = P'_{12}$ . Therefore the five possible 12-th tiles are all the same.

Now we find twelve tiles that are related as in Figure 8. If there are more tiles, then the additional tiles have to be glued to the existing twelve along some vertices or edges. This will either increase the degree of some vertex to be  $> 3$ , or cause some edge to be shared by more than two tiles. Therefore the twelve are all the tiles in the tiling. Moreover, if there are additional identifications among these twelve tiles, then the identification of vertices will increase the degree of some vertex to be  $> 3$ , the identification of edges will cause some edge to be shared by more than two tiles, and the identification of tiles will reduce the total number of tiles to be  $< 12$ . Since all these should not happen, there is no more additional identification. We conclude that Figure 8 gives the complete description of the tiling.  $\square$

### 3 Edge Congruent Tiling

We study the distribution of edge lengths in a spherical tiling by congruent pentagons. Since angles are ignored, we introduce the following concept.

**Definition 6.** Two polygons are *edge congruent*, if there is a correspondence between the edges, such that the adjacency is preserved and the edge lengths are preserved.

Two spherical (or planar) triangles with great arc (or straight line) edges are congruent if and only if they are edge congruent. However, this is no longer true for quadrilaterals and pentagons. Moreover, our results actually do not even require the edges to be great arcs (or straight).

**Proposition 7.** *Let  $a, b, c, \dots$ , denote distinct numbers. Then in a spherical tiling by edge congruent pentagons, the edge lengths of the tile must form one of the following five combinations:*

$$\begin{aligned} a^5 &= \{a, a, a, a, a\}, & a^4b &= \{a, a, a, a, b\}, \\ a^3b^2 &= \{a, a, a, b, b\}, & a^3bc &= \{a, a, a, b, c\}, & a^2b^2c &= \{a, a, b, b, c\}. \end{aligned}$$

*Proof.* By purely numerical consideration, there are seven possible combinations of five numbers. What we need to show is that the combinations

$$a^2bcd = \{a, a, b, c, d\}, \quad abcde = \{a, b, c, d, e\},$$

are impossible. Up to symmetry, all the possible arrangements of these two combinations are listed in Figure 10.

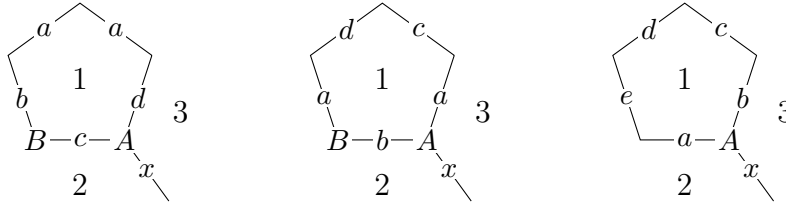


Figure 10: Impossible edge length arrangements.

In the first arrangement, by Lemma 3, one of the vertices  $A$  or  $B$  must have degree 3. By symmetry, we may assume  $\deg A = 3$  without loss of generality. Let  $x$  be the only other edge at  $A$  besides  $c$  and  $d$ . Then we have tiles  $P_2$  and  $P_3$  described in the picture. The edge  $x$  is adjacent to an edge of length  $c$  in  $P_2$ . Since  $P_2$  is edge congruent to  $P_1$ , and the edges adjacent to the unique edge of length  $c$  in  $P_1$  are the edges of lengths  $b$  and  $d$ . Therefore, we have  $x = b$  or  $x = d$  as far as the edge length is concerned. Similarly, the adjacency of  $x$  and  $d$  in  $P_3$  leads to  $x = a$  or  $x = c$  as far as the edge length is concerned. Since  $a, b, c, d$  are distinct, we get a contradiction.

We have been very careful with the wording in the argument above. We say “an edge of length  $c$ ” instead of “the edge  $c$ ” because this allows the possibility that the pentagon may have several edges of the same length  $c$ . This may happen when  $a, b, c, \dots$  are not assumed to be distinct. Since distinct letters indeed denote distinct values for the length in the current proposition, there is actually no confusion about the wording “the edge  $c$ ”. The wording in the subsequent discussion will be more casual.

In the second arrangement, by Lemma 3 and symmetry, we may still assume  $\deg A = 3$ . Then we may introduce  $x, P_2, P_3$  as before. By considering  $x$  in  $P_2$  and  $P_3$ , we see that  $x$  is adjacent to  $a$  and  $b$ . Since there is no such edge in  $P_1$ , we get a contradiction.

In the third arrangement, we may again assume  $\deg A = 3$  and introduce  $x, P_2, P_3$ . Then  $x$  is adjacent to both  $a$  and  $b$  from the viewpoint of  $P_2$  and  $P_3$ , yet there is no such edge in  $P_1$ .  $\square$

**Proposition 8.** *If a spherical tiling by edge congruent pentagons contains a tile with all vertices having degree 3, then the edge lengths must be arranged in one of the five ways in Figure 11.*

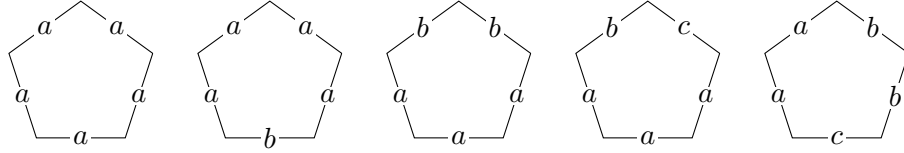


Figure 11: Edge length combinations when all vertices have degree 3.

*Proof.* By purely numerical consideration, the five possible edge length combinations  $a^5, a^4b, a^3b^2, a^3bc, a^2b^2c$  in Proposition 7 have respectively 1, 1, 2, 2, 3 possible arrangements. Besides the five in Figure 11, the remaining four are given in Figure 12. We need to argue that these four are impossible.

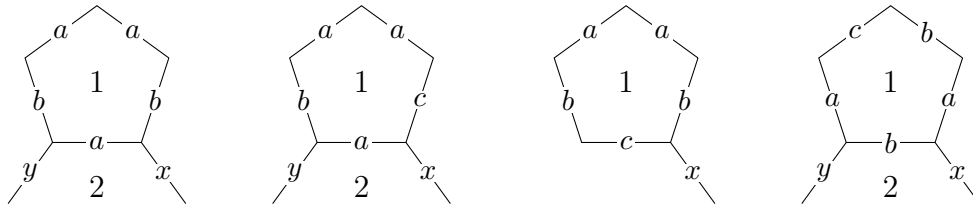


Figure 12: Impossible edge length arrangements.

In the first arrangement,  $x$  is adjacent to  $a$  and  $b$ . Since the only such edge in  $P_1$  is  $a$ , we get  $x = a$  (as far as the length is concerned). By the same reason, we also get  $y = a$ . Then in  $P_2$ , we get a row  $xya$  of three consecutive edges of the same length  $a$ . Since there is no such row in  $P_1$ , we get a contradiction.

In the second arrangement,  $x$  is adjacent to  $c$ , and  $y$  is adjacent to  $b$ . By the adjacency in  $P_1$ , we have  $x = y = a$ . Then the edge row  $xy = aaa$  appears in  $P_2$  but not in  $P_1$ . We get a contradiction.

In the third arrangement,  $x$  is adjacent to  $b$  and  $c$ . Since there is no such edge in  $P_1$ , we get a contradiction.

In the fourth arrangement,  $x$  and  $y$  are adjacent to  $a$  and  $b$ . By the adjacency in  $P_1$ , we have  $x = y = c$ . Then  $c$  appears twice in  $P_2$  but only once in  $P_1$ . We get a contradiction.  $\square$

## 4 Classification of Edge Congruent Tiling

The discussion in Section 3 is local, in the sense that it is not concerned with the global combinatorial structure given by Propositions 5. In this section, we discuss how the classification in Proposition 8 can match the global structure.

We denote the tiles in Figure 8 by  $P_1, P_2, \dots, P_{12}$ . We denote by  $E_{ij}$  the edge shared by the tiles  $P_i, P_j$ , and by  $V_{ijk}$  the vertex shared by  $P_i, P_j, P_k$ .

We also name an edge by its length and a vertex by the length of the edges at the vertex. For example, in Figure 2, the edge  $E_{12}$  is a  $b$ -edge, and the edge  $E_{13}$  is an  $a$ -edge. Moreover, the vertex  $V_{123}$  is an  $abc$ -vertex, and the vertex  $V_{134}$  is an  $a^3$ -vertex. The tiling has total of 12  $a$ -edges, 12  $b$ -edges, 6  $c$ -edges, 4  $a^3$ -vertices, 4  $b^3$ -vertices, and 12  $abc$ -vertices. We denote the *edgewise vertex combination* of the tiling by  $\{4a^3, 4b^3, 12abc\}$ .

For the edge length combination  $a^5$  (the first pentagon in Figure 11), we basically assign  $a$  to all edges in Figure 8. All vertices are  $a^3$ -vertices, and the edgewise vertex combination is  $\{20a^3\}$ . This can be realized by the regular dodecahedron tiling.

**Proposition 9.** *The spherical tilings by 12 edge congruent pentagons with edge lengths  $a, a, a, a, b$ , where  $a \neq b$ , are given by the tilings in Figure 13 up to symmetries. The unlabeled edges have length  $a$ .*

In proving the main theorem, we will not make use of the whole classification. The only fact we will use is the edgewise vertex combination  $\{8a^3, 12a^2b\}$  derived in the proof.

*Proof.* We need to assign 6  $b$ -edges to Figure 8, such that each tile has exactly one  $b$ -edge. In such an assignment, there is no  $ab^2$ -vertex and no  $b^3$ -vertex. So the edgewise vertex combination is  $\{ma^3, na^2b\}$ . On the other hand,



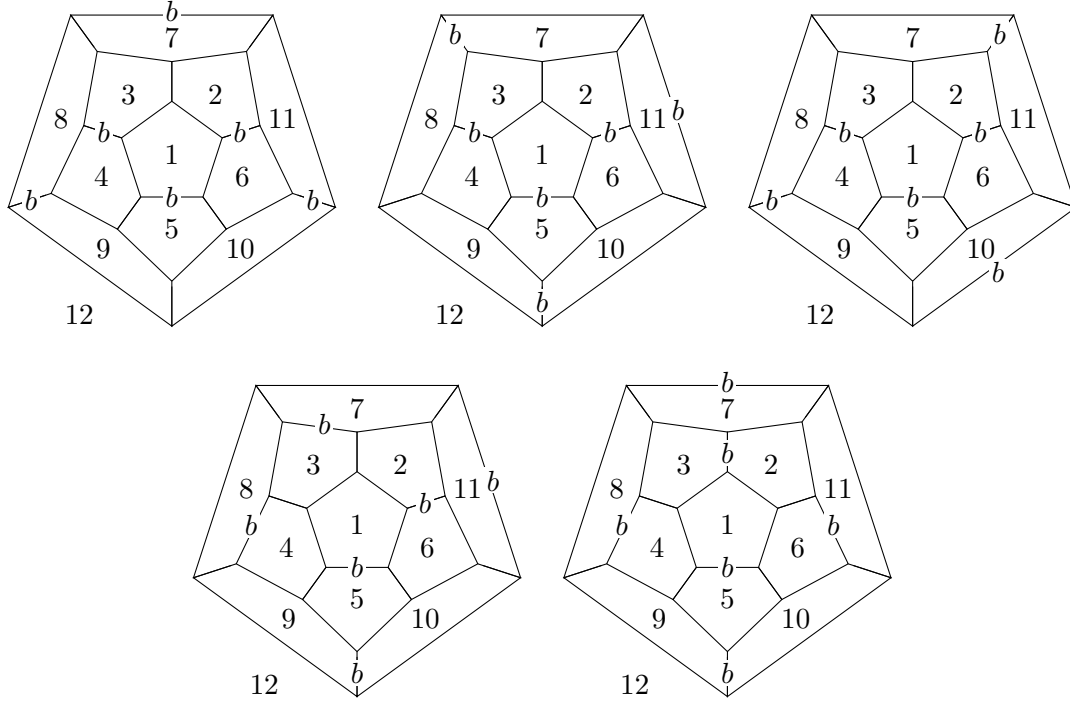


Figure 13: Tilings for the edge length combination  $a^4b$ .

$\frac{1}{5}$  of all the 30 edges are  $b$ -edges. Therefore there are  $N_a = 24$   $a$ -edges and  $N_b = 6$   $b$ -edges. Then from the edgewise vertex combination, we get  $3m + 2n = 2N_a = 48$  and  $n = 2N_b = 12$ . The solution is  $m = 8$  and  $n = 12$ .

For any tile  $P$  in Figure 8, the tile  $P$  and the five tiles around  $P$  form a tiling  $T_P$  of a “half sphere” (with wiggled boundary). Moreover, the remaining six tiles also form a tiling of the complementary “half sphere”. In fact,  $P$  has an “antipodal tile”  $P'$  (the antipodal of  $P_1$  is  $P_{12}$ , for example), and the remaining six tiles form the tiling  $T_{P'}$ .

We say an assignment of  $b$ -edges is *splitting*, if there is a half sphere tiling  $T_P$ , such that if a  $b$ -edge is shared by  $P_i$  and  $P_j$ , then  $P_i \in T_P$  if and only if  $P_j \in T_P$ . This implies that  $P_i \in T_{P'}$  if and only if  $P_j \in T_{P'}$ . We also call the tile  $P$  a *splitting center*.

A splitting assignment is then the union of two independent assignments of 3  $b$ -edges to  $T_P$  and the other 3  $b$ -edges to  $T_{P'}$ . In this case, it is easy to see that the choice of the  $b$ -edge for  $P$  completely determines the other two  $b$ -edges in  $T_P$ . See  $T_{P_1}$  in the first three tilings in Figure 13. Up to

symmetry, we have three possible combinations of the choice of  $b$ -edges for  $P_1$  and  $P_{12}$ , which give the first three tilings in Figure 13. They are not equivalent because they have respectively 6, 2 and 4 splitting centers.

Next we search for non-splitting assignments. Assume there are edges connecting two  $b$ -edges. Without loss of generality, we may assume  $E_{15} = E_{26} = b$  as in the fourth tiling in Figure 13. Since  $P_1$  and  $P_6$  are not splitting centers, we get  $E_{34} \neq b$  and  $E_{1011} \neq b$ . If  $E_{711} = b$ , then the only possible  $b$ -edge of  $P_3$  is  $E_{38} = b$ . This further implies that the  $b$ -edge of  $P_4$  is  $E_{49} = b$ . Then  $P_4$  becomes a splitting center. Therefore  $E_{711} \neq b$ , and the only possible  $b$ -edge of  $P_{11}$  is  $E_{1112} = b$ . This further implies successively that the  $b$ -edge of  $P_{10}$  is  $E_{910}$ , the  $b$ -edge of  $P_4$  is  $E_{48}$ , and  $E_{37} = b$ .

Finally we look for non-splitting assignments such that no edge connects two  $b$ -edges. Without loss of generality, we may assume  $E_{15} = b$ . Since  $E_{26}$ ,  $E_{34}$ ,  $E_{49}$ ,  $E_{610}$  are connected to  $E_{15}$  by one edge, they are not  $b$ -edges. This implies that  $E_{48} = E_{611} = b$ . Then the 8 edges connected to  $E_{48}$  and  $E_{611}$  by one edge cannot be  $b$ -edges. This is enough for us to determine the remaining 3  $b$ -edges and get the fifth tiling in Figure 13.  $\square$

**Proposition 10.** *The spherical tiling by 12 edge congruent pentagons with edge lengths  $a, a, a, b, b$ , where  $a \neq b$ , is given by the left of Figure 14 up to symmetries.*

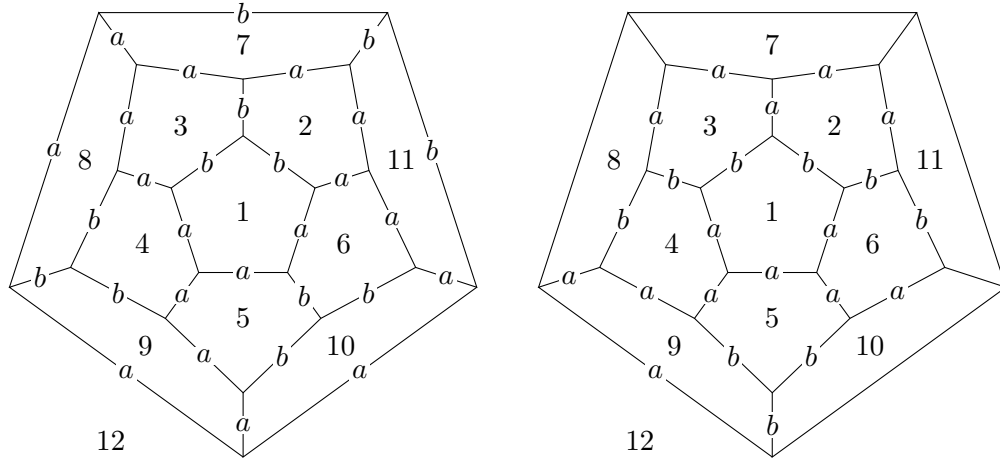


Figure 14: Tilings for the edge length combination  $a^3b^2$ .

*Proof.* By Proposition 8, we may start with  $P_1$ , with the edges arranged the same as the third in Figure 11. The length of  $E_{23}$  is either  $a$  or  $b$ .

The case  $E_{23} = b$  is described on the left of Figure 14. From  $E_{12} = E_{23} = b$ , we see the other three edges of  $P_2$  have length  $a$ . By the same reason, we get all the edge lengths of  $P_3$ . Then among  $E_{78}$  and  $E_{7\overline{11}}$ , one is  $a$  and one is  $b$ . By symmetry, we may assume  $E_{78} = a$  and  $E_{7\overline{11}} = b$ . Then  $E_{7\overline{12}} = b$ . From  $E_{2\overline{11}} = a$ ,  $E_{7\overline{11}} = b$ , we get all the edges of  $P_{11}$ . From  $E_{16} = E_{26} = E_{6\overline{11}} = a$ , we get all the edges of  $P_6$ . From  $E_{6\overline{10}} = b$ ,  $E_{\overline{10}\overline{11}} = a$ , we get all the edges of  $P_{10}$ . From  $E_{56} = E_{5\overline{10}} = b$ , we get all the edges of  $P_5$ . From  $E_{14} = E_{34} = E_{45} = a$ , we get all the edges of  $P_4$ . From  $E_{49} = b$ ,  $E_{59} = a$ , we get all the edges of  $P_9$ . From  $E_{38} = a$ ,  $E_{48} = b$ , we get all the edges of  $P_8$ . This concludes all the edge lengths.

The case  $E_{23} = a$  is the right of Figure 14. We can immediately get all the edges of  $P_2$ ,  $P_3$ . Afterwards, we get all the edges of  $P_4$ ,  $P_6$ . Now we know three edges of  $P_5$  to be  $a$ , so that the other two edges are  $b$ . Then we get all the edges of  $P_9$ , and find that  $P_8$  has one  $b$ -edge adjacent to two  $a$ -edges, a contradiction.  $\square$

**Proposition 11.** *The spherical tiling by 12 edge congruent pentagons cannot have edge lengths  $a, a, a, b, c$ , for distinct  $a, b, c$ .*

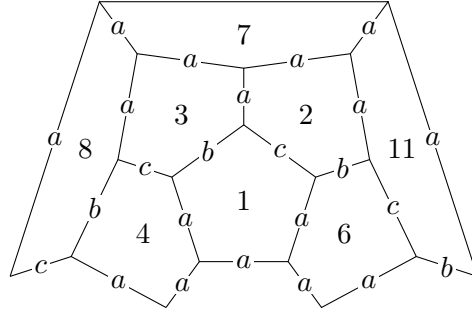


Figure 15: Tilings for the edge length combination  $a^2bcd$ .

*Proof.* By Proposition 8, we may start with  $P_1$ , with the edges arranged the same as the fourth in Figure 11. Since  $E_{23}$  is adjacent to  $b$  and  $c$ , we get  $E_{23} = a$ . Then we successively get all the edges of  $P_2$ ,  $P_3$ ,  $P_4$ ,  $P_6$ ,  $P_8$ ,  $P_{11}$ . Now we have four  $a$ -edges in  $P_7$ , a contradiction.  $\square$

**Proposition 12.** *The spherical tiling by 12 edge congruent pentagons with edge lengths  $a, a, b, b, c$ , where  $a, b, c$  are distinct, is given by the left of Figure 16 up to symmetries.*

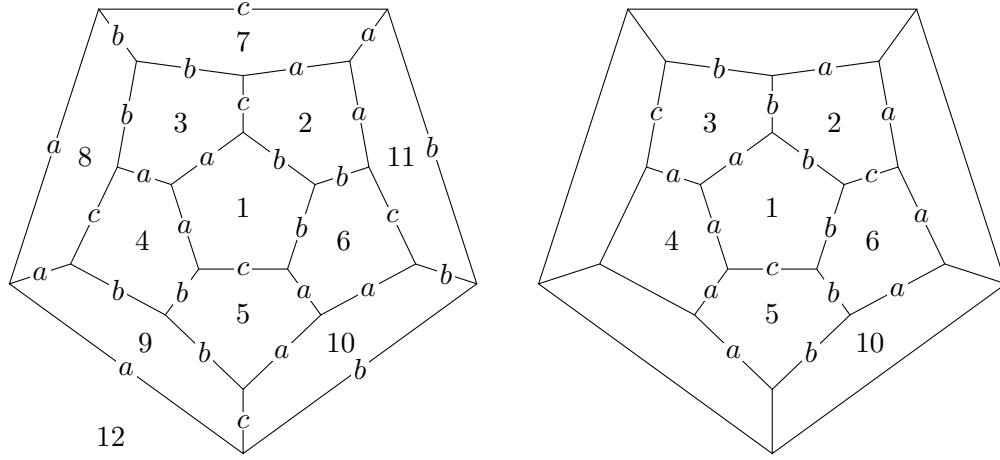


Figure 16: Tilings for the edge length combination  $a^2b^2c$ .

*Proof.* By Proposition 8, we may start with  $P_1$ , with the edges arranged the same as the fifth in Figure 11. Since  $E_{45}$  is adjacent to  $c$ , its length is either  $a$  or  $b$ .

The case  $E_{45} = b$  is the left of Figure 16. From  $E_{14} = a$ ,  $E_{15} = c$ ,  $E_{45} = b$ , we get all the edges of  $P_4$ ,  $P_5$ . Then we further get all the edges of  $P_6$ . Since we cannot have three  $a$ -edges or three  $b$ -edges in a tile, we must have  $E_{23} = c$  and then get all the edges of  $P_2$ ,  $P_3$ . By the same reason, we get  $E_{9\overline{10}} = c$  and all the edges of  $P_9$ ,  $P_{10}$ . Then we may get all the edges of  $P_7$ ,  $P_8$ ,  $P_{11}$ . This concludes all the edge lengths.

The case  $E_{45} = a$  is the right of Figure 16. We immediately get all the edges of  $P_5$ . Then in  $P_6$ , either  $E_{26}$  or  $E_{6\overline{10}}$  is  $c$ . The two assumptions are actually equivalent by symmetry, so that we may assume  $E_{26} = c$  without loss of generality. Then we get all the edges of  $P_2$ ,  $P_6$ . After further getting all the edges of  $P_3$ , we find three  $a$ -edges in  $P_4$ , a contradiction.  $\square$

## 5 Angle Congruent Tiling

Now we turn to the distribution of angles in a spherical tiling by congruent pentagons. Since edge lengths are ignored, we introduce the following concept.

**Definition 13.** Two polygons are *angle congruent*, if there is a correspondence between the edges, such that the adjacency is preserved and the angles between the adjacent edges are preserved.

Two spherical triangles with great arc edges are congruent if and only if they are angle congruent. Two planar triangles with straight line edges are similar if and only if they are angle congruent. However, this is no longer true for quadrilaterals and pentagons.

In the regular dodecahedron tiling, all the angles are

$$\alpha = \frac{2\pi}{3}.$$

In what follows, we will always reserve  $\alpha$  for this special angle, and let  $\beta, \gamma, \dots$  be angles not equal to  $\alpha$ . In this section, we may occasionally allow some from  $\beta, \gamma, \dots$  to be equal. In the later sections, we will assume  $\beta, \gamma, \dots$  to be distinct.

**Lemma 14.** *In a spherical tiling by 12 angle congruent pentagons, the sum of five angles in a tile is  $\frac{10\pi}{3} = 5\alpha$ .*

In case the edges are great arcs, the lemma is a consequence of the formula for the area of a spherical pentagon. However, the lemma holds even if the edges are not great arcs.

*Proof.* The sum of angles at each vertex is  $2\pi$ . There are 20 vertices in such a tiling, so that the total sum is  $40\pi$ . On the other hand, the sum  $\Sigma$  of five angles in a tile is the same for all tiles because of the angle congruence. Since there are 12 tiles, we get  $40\pi = 12\Sigma$ . This implies  $\Sigma = \frac{10\pi}{3}$ .  $\square$

In Section 4, we named a vertex by the lengths of edges at the vertex. Similarly, we can also name a vertex by the angles at the vertex. For example, in Figure 2,  $V_{123}$  is a  $\beta\gamma\delta$ -vertex, and  $V_{126}$  is an  $\alpha^3$ -vertex. The tiling has 8  $\alpha^3$ -vertices and 12  $\beta\gamma\delta$ -vertices, and has the *anglewise vertex combination*

$\{8\alpha^3, 12\beta\gamma\delta\}$ . We also say that  $\alpha^3$  and  $\beta\gamma\delta$  are vertices in the tiling, and the other combinations are not vertices.

We say a vertex is of  $\alpha\beta\gamma$ -type, if the vertex is an  $\alpha\beta'\gamma'$ -vertex, where  $\beta'$  and  $\gamma'$  may or may not be  $\beta$  and  $\gamma$ . Thus an  $\alpha\beta\gamma$ -type vertex can be an  $\alpha\beta\gamma$ -vertex, or an  $\alpha\beta\delta$ -vertex, or an  $\alpha\gamma\delta$ -vertex, etc. From purely symbolical viewpoint (recall that the symbol  $\alpha$  is special), a degree 3 vertex must be one of the following types:

$$\alpha^3, \alpha^2\beta, \alpha\beta^2, \alpha\beta\gamma, \beta^3, \beta^2\gamma, \beta\gamma\delta.$$

**Lemma 15.** *A vertex of degree 3 must be one of the following types:*

1.  $\alpha^3$ .
2.  $\alpha\beta\gamma$  with  $\beta \neq \gamma$ .
3.  $\beta^2\gamma$  with  $\beta \neq \gamma$ .
4.  $\beta\gamma\delta$  with  $\beta, \gamma, \delta$  distinct.

*Proof.* We need to show that, if  $\alpha \neq \beta$ , then  $\alpha^2\beta$ ,  $\alpha\beta^2$ ,  $\beta^3$  are not allowed. If  $\alpha^2\beta$  is a vertex, then  $2\alpha + \beta = 2\pi = 3\alpha$ , which implies  $\beta = \alpha$ , a contradiction. If  $\alpha\beta^2$  is a vertex, then  $\alpha + 2\beta = 2\pi = 3\alpha$ , which again implies  $\beta = \alpha$ . If  $\beta^3$  is a vertex, then  $3\beta = 2\pi = 3\alpha$ , which still implies  $\beta = \alpha$ .  $\square$

**Lemma 16.** *Suppose  $\beta, \gamma, \delta, \epsilon$  are distinct angles.*

1. *If  $\beta^2\gamma$  is a vertex, then  $\beta^2\delta$ ,  $\beta\gamma^2$ ,  $\beta\gamma\delta$  are not vertices.*
2. *If  $\beta\gamma\delta$  is a vertex, then  $\beta^2\gamma$ ,  $\beta\gamma\epsilon$  are not vertices.*

The lemma actually allows some angles to be  $\alpha$ . Moreover, we get more exclusions by symmetry. For example, if  $\beta\gamma\delta$  is a vertex, then the following cannot be vertices:

$$\beta^2\gamma, \beta^2\delta, \beta\gamma^2, \beta\delta^2, \gamma^2\delta, \gamma\delta^2, \beta\gamma\epsilon, \beta\delta\epsilon, \gamma\delta\epsilon.$$

*Proof.* If both  $\beta^2\gamma$  and  $\beta^2\delta$  are vertices, then  $2\beta + \gamma = 3\alpha$  and  $2\beta + \delta = 3\alpha$ . This implies  $\gamma = \delta$ , a contradiction.

If both  $\beta^2\gamma$  and  $\beta\gamma^2$  are vertices, then  $2\beta + \gamma = 3\alpha$  and  $\beta + 2\gamma = 3\alpha$ . This implies  $\beta = \gamma$ , a contradiction.

If both  $\beta^2\gamma$  and  $\beta\gamma\delta$  are vertices, then  $2\beta + \gamma = 3\alpha$  and  $\beta + \gamma + \delta = 3\alpha$ . This implies  $\beta = \delta$ , again a contradiction.

The proof of the second statement is similar.  $\square$

**Proposition 17.** *Let  $\alpha = \frac{2}{3}\pi$  and let  $\alpha, \beta, \gamma, \dots$  denote distinct angles. Then a spherical tiling by 12 angle congruent pentagons must have one of the following (unordered) angle combinations:*

1.  $\alpha^5$ : *The angles in the pentagon are  $\alpha, \alpha, \alpha, \alpha, \alpha$ . The anglewise vertex combination is  $\{20\alpha^3\}$ .*
2.  $\alpha^3\beta\gamma$ : *The angles in the pentagon are  $\alpha, \alpha, \alpha, \beta, \gamma$ , satisfying  $\beta + \gamma = 2\alpha$ . The anglewise vertex combination is  $\{8\alpha^3, 12\alpha\beta\gamma\}$ .*
3.  $\alpha^2\beta^2\gamma$ : *The angles in the pentagon are  $\alpha, \alpha, \beta, \beta, \gamma$ , satisfying  $2\beta + \gamma = 3\alpha$ . The anglewise vertex combination is  $\{8\alpha^3, 12\beta^2\gamma\}$ .*
4.  $\alpha^2\beta\gamma\delta$ : *The angles in the pentagon are  $\alpha, \alpha, \beta, \gamma, \delta$ , satisfying  $\beta + \gamma + \delta = 3\alpha$ . The anglewise vertex combination is  $\{8\alpha^3, 12\beta\gamma\delta\}$ .*
5.  $\alpha\beta\gamma\delta\epsilon$ : *The angles in the pentagon are  $\alpha, \beta, \gamma, \delta, \epsilon$ , satisfying  $\gamma = 3\alpha - 2\beta$ ,  $\delta = 2\alpha - \beta$ ,  $\epsilon = 2\beta - \alpha$ . There are three possible anglewise vertex combinations:  $\{4\alpha\beta\delta, 8\alpha\gamma\epsilon, 4\beta^2\gamma, 4\delta^2\epsilon\}$ ,  $\{\alpha^3, 2\alpha\beta\delta, 7\alpha\gamma\epsilon, 5\beta^2\gamma, 5\delta^2\epsilon\}$ ,  $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$ .*

The fourth case degenerates into the second (when  $\delta = \alpha$ ) and the third (when  $\delta = \beta$ ) cases, which further degenerate into the first case. The tiling  $T_5$  in Figure 1 and Figure 2 is the fourth case and its degenerate cases. The tilings  $T_1, T_2, T_3, T_4$  in Figure 1 belong to the third anglewise vertex combination in the fifth case.

*Proof.* We divide the proof by considering how many times the angle  $\alpha$  appears in the pentagon.

**Case 1:** If the angles in the pentagon are  $\alpha, \alpha, \alpha, \alpha, \beta$ , then Lemma 14 tells us  $4\alpha + \beta = 5\alpha$ , so that the angles are really  $\alpha, \alpha, \alpha, \alpha, \alpha$ . This is the first case in the proposition.

**Case 2:** The angles in the pentagon are  $\alpha, \alpha, \alpha, \beta, \gamma$ , with  $\beta, \gamma \neq \alpha$ .

Lemma 14 becomes  $\beta + \gamma = 2\alpha$ . If  $\beta^2\gamma$  is a vertex, then  $2\beta + \gamma = 2\pi = 3\alpha$ , and we get  $\beta = \gamma = \alpha$ , a contradiction. By the similar reason,  $\beta\gamma^2$  is not a vertex. By Lemma 15, the only possible vertices are  $\alpha^3$  and  $\alpha\beta\gamma$ .

Let  $\{m\alpha^3, n\alpha\beta\gamma\}$  be the anglewise vertex combination. Since 12 tiles with angles  $\alpha, \alpha, \alpha, \beta, \gamma$  in each tile have total of 36 angle  $\alpha$ , 12 angle  $\beta$  and 12 angle  $\gamma$ , we get  $3m + n = 36$  and  $n = 12$ . Therefore  $m = 8$  and  $n = 12$ . This is the second case of the proposition.

**Case 3:** The angles in the pentagon are  $\alpha, \alpha, \beta, \gamma, \delta$ , with  $\beta, \gamma, \delta \neq \alpha$ . Note that  $\beta, \gamma, \delta$  are not yet assumed to be distinct.

Lemma 14 becomes  $\beta + \gamma + \delta = 3\alpha$ . If  $\alpha\beta\gamma$  is a vertex, then  $\alpha + \beta + \gamma = 3\alpha$ , and we get  $\delta = \alpha$ , a contradiction. The argument actually shows that there is no  $\alpha\beta\gamma$ -type vertex.

If  $\beta^2\gamma$  is a vertex, then  $2\beta + \gamma = 3\alpha$  and  $\beta = 3\alpha - \beta - \gamma = \delta$ . Thus the  $\beta^2\gamma$ -vertex is really a  $\beta\gamma\delta$ -vertex. The argument actually applies to any  $\beta^2\gamma$ -type vertex. By Lemma 15, we conclude that any vertex is either an  $\alpha^3$ -vertex or a  $\beta\gamma\delta$ -vertex.

Let  $\{m\alpha^3, n\beta\gamma\delta\}$  be the anglewise vertex combination. Without loss of generality, we may further assume that  $\gamma$  is different from  $\alpha, \beta, \delta$ . Since 12 tiles with angles  $\alpha, \alpha, \beta, \gamma, \delta$  have total of 24 angle  $\alpha$  and 12 angle  $\gamma$ , we get  $3m = 24$ ,  $n = 12$ . Therefore  $m = 8$  and  $n = 12$ . Depending on whether  $\beta = \delta$ , we get the third and the fourth cases of the proposition.

**Case 4:** The angles in the pentagon are  $\alpha, \beta, \gamma, \delta, \epsilon$ , with  $\beta, \gamma, \delta, \epsilon \neq \alpha$ . Again  $\beta, \gamma, \delta, \epsilon$  are not yet assumed to be distinct.

Lemma 14 becomes  $\beta + \gamma + \delta + \epsilon = 4\alpha$ . We further divide the case by considering whether some among  $\beta, \gamma, \delta, \epsilon$  are equal.

**Case 4.1:**  $\beta = \gamma = \delta = \epsilon$ .

Lemma 14 becomes  $4\beta = 4\alpha$ , and we get  $\beta = \alpha$ , a contradiction.

**Case 4.2:**  $\beta = \delta = \epsilon \neq \gamma$ .

Lemma 14 becomes  $3\beta + \gamma = 4\alpha$ . There is no  $\alpha\beta\gamma$ -vertex because this implies  $\alpha + \beta + \gamma = 3\alpha$ , and we get  $\beta = \gamma = \alpha$ . Similar argument shows that  $\beta^2\gamma$ ,  $\beta\gamma^2$  are not vertices. By Lemma 15, the only vertex is  $\alpha^3$ , which is impossible.

**Case 4.3:**  $\beta = \delta \neq \gamma = \epsilon$ .

Lemma 14 becomes  $\beta + \gamma = 2\alpha$ . An argument similar to Case 4.2 shows that  $\beta^2\gamma$ ,  $\beta\gamma^2$  are not vertices. By Lemma 15, the only vertices are  $\alpha^3$  and  $\alpha\beta\gamma$ , so that the anglewise vertex combination is  $\{m\alpha^3, n\alpha\beta\gamma\}$ . Since 12 tiles with angles  $\alpha, \beta, \beta, \gamma, \gamma$  have total of 12 angle  $\alpha$  and 24 angle  $\beta$ , we get  $3m + n = 12$  and  $n = 24$ . The system has no non-negative solution, a contradiction.

**Case 4.4:**  $\beta = \epsilon$ , and  $\beta, \gamma, \delta$  are distinct.

Similar to Case 4.2,  $\alpha\gamma\delta$ ,  $\beta^2\gamma$ ,  $\beta^2\delta$ ,  $\beta\gamma\delta$  are not vertices. By the symmetry of  $\gamma$  and  $\delta$  in the current case and the exclusion between  $\alpha\beta\gamma$  and  $\alpha\beta\delta$  (see



Lemma 16), we may assume that the only possible  $\alpha\beta\gamma$ -type vertex is  $\alpha\beta\gamma$ . Then by Lemma 15, the anglewise vertex combination is

$$\{m_1\alpha^3, m_2\alpha\beta\gamma, n_1\beta\gamma^2, n_2\beta\delta^2, k_1\gamma^2\delta, k_2\gamma\delta^2\}.$$

Since 12 tiles with angles  $\alpha, \beta, \beta, \gamma, \delta$  have total of 12 angle  $\alpha$ , 24 angle  $\beta$ , 12 angle  $\gamma$  and 12 angle  $\delta$ , we get

$$\begin{aligned} 3m_1 + m_2 &= 12, \\ m_2 + n_1 + n_2 &= 24, \\ m_2 + 2n_1 + 2k_1 + k_2 &= 12, \\ 2n_2 + k_1 + 2k_2 &= 12. \end{aligned}$$

Adding the third and the fourth equations together, we get  $m_2 + 2(n_1 + n_2) + 3(k_1 + k_2) = 24$ . Compared with the second equation, we get  $n_1 = n_2 = k_1 = k_2 = 0$ . Then the second equation becomes  $m_2 = 24$ , contradicting to the first equation.

**Case 4.5:**  $\beta, \gamma, \delta, \epsilon$  distinct.

Similar to Case 4.2, there is no  $\beta\gamma\delta$ -type vertex. By Lemma 15, the only possible types are  $\alpha^3, \alpha\beta\gamma, \beta^2\gamma$ . We further divide into two subcases.

**Case 4.5.1:**  $\beta$  appears in two vertices not involving  $\alpha$  and with different angle combinations.

Without loss of generality, we may assume the two vertices are  $\beta^i\gamma^{3-i}$  and  $\beta^j\delta^{3-j}$ ,  $0 < i, j < 3$ . By Lemma 16, we have  $i \neq j$ . Therefore up to symmetry, we may assume  $\beta^2\gamma, \beta\delta^2$  are vertices. Solving  $2\beta + \gamma = \beta + 2\delta = 3\alpha$  together with  $\beta + \gamma + \delta + \epsilon = 4\alpha$  from Lemma 14, we get

$$3\beta - 2\epsilon = \alpha, \quad 3\gamma + 4\epsilon = 7\alpha, \quad 3\delta + \epsilon = 4\alpha.$$

Now the angle  $\epsilon$  must appear at some vertex. Since there is no  $\beta\gamma\delta$ -type vertex, by Lemmas 15 and 16, one of  $\alpha\beta\epsilon, \alpha\gamma\epsilon, \alpha\delta\epsilon, \gamma^2\epsilon, \delta^2\epsilon, \gamma\epsilon^2, \delta\epsilon^2$  must appear as the vertex involving  $\epsilon$ . If  $\alpha\beta\epsilon$  is a vertex, then  $\delta + \epsilon = 2\alpha$  and  $3\beta - 2\epsilon = \alpha$  imply  $\beta = \epsilon$ , a contradiction. All other possibilities lead to similar contradictions.

**Case 4.5.2:** The opposite of Case 4.5.1 and its symmetric permutations.

This means that, without loss of generality, we may assume that the only possible vertices not involving  $\alpha$  are  $\beta^2\gamma$  and  $\delta^2\epsilon$ . Then the anglewise vertex

combination is  $\alpha^3, \beta^2\gamma, \delta^2\epsilon$  together with some  $\alpha\beta\gamma$ -type vertices. By Lemma 16, we find only two possibilities:

$$\{m\alpha^3, n_1\alpha\beta\delta, n_2\alpha\gamma\epsilon, k_1\beta^2\gamma, k_2\delta^2\epsilon\}, \quad \{m\alpha^3, n_1\alpha\beta\epsilon, n_2\alpha\gamma\delta, k_1\beta^2\gamma, k_2\delta^2\epsilon\}.$$

In the first combination, counting the total number of each angle gives

$$3m + n_1 + n_2 = n_1 + 2k_1 = n_2 + k_1 = n_1 + 2k_2 = n_2 + k_2 = 12.$$

We get  $k_1 = k_2 = m + 4$ ,  $n_1 = 4 - 2m$ ,  $n_2 = 8 - m$ . In particular, we have  $n_2, k_1, k_2 > 0$ , so that  $\alpha\gamma\epsilon, \beta^2\gamma, \delta^2\epsilon$  are vertices, and we get

$$\gamma + \epsilon = 2\alpha, \quad 2\beta + \gamma = 3\alpha, \quad 2\delta + \epsilon = 3\alpha.$$

The solution is

$$\gamma = 3\alpha - 2\beta, \quad \delta = 2\alpha - \beta, \quad \epsilon = 2\beta - \alpha.$$

On the other hand, to maintain  $n_1 \geq 0$ , the possible values for  $m$  are 0, 1, 2, and we get the corresponding possible anglewise vertex combinations. These are the three possible combinations in the fifth case of the proposition.

The second combination gives

$$3m + n_1 + n_2 = n_1 + 2k_1 = n_2 + k_1 = n_2 + 2k_2 = n_1 + k_2 = 12.$$

The system has no non-negative solution.

**Case 5:** The angles in the pentagon are  $\beta, \gamma, \delta, \epsilon, \zeta$ , all different from  $\alpha$ . But the five angles are not yet assumed to be distinct.

Lemma 14 becomes  $\beta + \gamma + \delta + \epsilon + \zeta = 5\alpha$ . We further divide the case by considering whether some angles are equal.

**Case 5.1:**  $\beta = \gamma = \delta = \epsilon = \zeta$ .

Lemma 14 implies  $\beta = \alpha$ , a contradiction.

**Case 5.2:**  $\beta = \delta = \epsilon = \zeta \neq \gamma$ .

The vertices must be  $\beta^2\gamma$  or  $\beta\gamma^2$ . However, Lemma 14 becomes  $4\beta + \gamma = 5\alpha$ . If  $\beta^2\gamma$  is a vertex, then  $2\beta + \gamma = 3\alpha$ , and we get  $\beta = \gamma$ , a contradiction. The appearance of  $\beta\gamma^2$  leads to the same contradiction.

**Case 5.3:**  $\beta = \delta = \epsilon \neq \gamma = \zeta$ .

The vertices must be  $\beta^2\gamma$  or  $\beta\gamma^2$ . We get contradiction similar to Case 5.2.

**Case 5.4:**  $\beta = \epsilon = \zeta$ , and  $\beta, \gamma, \delta$  are distinct.

Lemma 14 becomes  $3\beta + \gamma + \delta = 5\alpha$ . This implies that  $\beta\gamma\delta$  is not a vertex, so that all vertices are  $\beta^2\gamma$ -type. Since there are three distinct angles, one angle must appear in two vertices with different angle combinations. By Lemma 16 and the symmetry between  $\gamma$  and  $\delta$ , we only need to consider both  $\beta^2\gamma, \beta\delta^2$  appear, or both  $\beta^2\gamma, \gamma^2\delta$  appear, or both  $\beta\gamma^2, \gamma\delta^2$  appear.

If  $\beta^2\gamma, \beta\delta^2$  are vertices, then  $2\beta + \gamma = 3\alpha$  and  $\beta + 2\delta = 3\alpha$ . Combined with  $3\beta + \gamma + \delta = 5\alpha$ , we get all angles equal, a contradiction. The other pairs of vertices lead to similar contradictions.

**Case 5.5:**  $\beta = \epsilon, \gamma = \zeta$ , and  $\beta, \gamma, \delta$  are distinct.

Lemma 14 becomes  $2\beta + 2\gamma + \delta = 5\alpha$ . This implies that  $\beta^2\delta, \gamma^2\delta, \beta\gamma\delta$  are not vertices. Since there are three distinct angles and all vertices are  $\beta^2\gamma$ -type, one angle must appear in two vertices with different angle combinations. By Lemma 16, the symmetry between  $\beta$  and  $\gamma$ , and the fact that  $\beta^2\delta$  and  $\gamma^2\delta$  are not vertices, we may assume that  $\beta^2\gamma, \beta\delta^2$  are vertices. This implies  $2\beta + \gamma = 3\alpha$  and  $\beta + 2\delta = 3\alpha$ . Combined with  $2\beta + 2\gamma + \delta = 5\alpha$ , we get all angles equal, a contradiction.

**Case 5.6:**  $\beta = \zeta$ , and  $\beta, \gamma, \delta, \epsilon$  are distinct.

Lemma 14 becomes  $2\beta + \gamma + \delta + \epsilon = 5\alpha$ . This implies that  $\gamma\delta\epsilon$  is not a vertex.

If there is a vertex of  $\beta\gamma\delta$ -type, then by symmetry, we may assume  $\beta\gamma\delta$  is a vertex. Then we get  $\beta + \gamma + \delta = 3\alpha$  and  $\beta + \epsilon = 5\alpha - (\beta + \gamma + \delta) = 2\alpha$ . This implies that  $\beta, \gamma, \delta$  cannot form  $\beta^2\gamma$ -type vertices, and  $\beta^2\epsilon, \beta\epsilon^2$  are not vertices. So the only possible vertices of  $\beta^2\gamma$ -type are  $\gamma^2\epsilon, \gamma\epsilon^2, \delta^2\epsilon, \delta\epsilon^2$ . Moreover, by Lemma 16, the appearance of  $\gamma^2\epsilon$  excludes  $\gamma\epsilon^2$  and  $\delta^2\epsilon$ . Then by the symmetry between  $\gamma$  and  $\delta$  in our current situation, we only need to consider the anglewise vertex combination  $\{n\beta\gamma\delta, k_1\gamma^2\epsilon, k_2\delta\epsilon^2\}$ . Counting the total number of each angle gives

$$n = 24, \quad n + 2k_1 = n + k_2 = k_1 + 2k_2 = 12.$$

The system has no non-negative solution.

So all vertices are of  $\beta^2\gamma$ -type, and the anglewise vertex combination is

$$\{m_1\beta^2\gamma, m_2\beta^2\delta, m_3\beta^2\epsilon, n_1\beta\gamma^2, n_2\beta\delta^2, n_3\beta\epsilon^2, k_1\gamma^2\delta, k_2\gamma^2\epsilon, k_3\gamma\delta^2, k_4\gamma\epsilon^2, k_5\delta^2\epsilon, k_6\delta\epsilon^2\}.$$

Counting the total of  $\beta$  gives  $2(m_1 + m_2 + m_3) + n_1 + n_2 + n_3 = 24$ .

If  $m_1 = m_2 = m_3 = 0$  and  $n_1 > 0$ , then by Lemma 16, we get  $n_2 = n_3 = 0$  and  $n_1 = 24$ . Therefore the total number of  $\gamma$  is  $\geq 2n_1 = 48$ , a contradiction.

Similar argument rules out  $n_2 > 0$  and  $n_3 > 0$ . Therefore  $m_1, m_2, m_3$  cannot be all 0, and we may assume  $m_1 > 0$  without loss of generality. By Lemma 16, this implies  $m_2 = m_3 = n_1 = k_3 = k_4 = 0$ , and we get  $2m_1 + n_2 + n_3 = 24$ .

If  $n_2 = n_3 = 0$ , then we get  $m_1 = 12$ . Thus the 12  $\beta^2\gamma$ -vertices already contain all 12 appearances of the angle  $\gamma$ . Therefore  $\gamma$  does not appear in any other vertex, and we are left with the anglewise vertex combination  $\{12\beta^2\gamma, k_5\delta^2\epsilon, k_6\delta\epsilon^2\}$ . By counting the total number of  $\delta$  and  $\epsilon$ , we get  $2k_5 + k_6 = k_5 + 2k_6 = 12$ . This implies  $k_5 = k_6 = 4 > 0$ . Therefore both  $\delta^2\epsilon$  and  $\delta\epsilon^2$  are vertices, a contradiction to Lemma 16.

So we may assume  $n_2 > 0$ . By Lemma 16, we get  $n_3 = k_5 = 0$ , so that the anglewise vertex combination becomes  $\{m_1\beta^2\gamma, n_2\beta\delta^2, k_1\gamma^2\delta, k_2\gamma^2\epsilon, k_6\delta\epsilon^2\}$ . By counting the total number of each angle, we get

$$2m_1 + n_2 = 24, \quad m_1 + 2k_1 + 2k_2 = 2n_2 + k_1 + k_6 = k_2 + 2k_6 = 12.$$

By Lemma 16, we must have either  $k_1 = 0$  or  $k_2 = 0$ . If  $k_1 = 0$ , the system gives  $k_2 = \frac{4}{5}$ . If  $k_2 = 0$ , the system gives  $k_1 = \frac{2}{3}$ . Since the numbers are not integers, we get a contradiction.

**Case 5.7:**  $\beta, \gamma, \delta, \epsilon, \zeta$  are distinct.

**Case 5.7.1:** There are  $\beta\gamma\delta$ -type vertices.

We may assume  $\beta\gamma\delta$  is a vertex. This implies  $\beta + \gamma + \delta = 3\alpha$ , so that  $\beta, \gamma, \delta$  cannot form  $\beta^2\gamma$ -type vertices. By Lemma 14, we have  $\epsilon + \zeta = 5\alpha - (\beta + \gamma + \delta) = 2\alpha$ , so that  $\epsilon, \zeta$  cannot appear at the same vertex. By Lemma 16, the only  $\beta\gamma\delta$ -type vertex is  $\beta\gamma\delta$ , and the only way for  $\epsilon, \zeta$  to appear is to get combined with angles from  $\beta, \gamma, \delta$ .

If  $\epsilon, \zeta$  can be combined with the same angle from  $\beta, \gamma, \delta$ , then up to symmetry, we may assume  $\beta\gamma\delta, \beta^2\epsilon, \beta\zeta^2$  are vertices. We get  $\beta + \gamma + \delta = 2\beta + \epsilon = \beta + 2\zeta = 3\alpha$ . Together with  $\epsilon + \zeta = 2\alpha$ , we get  $\delta = \alpha$ , a contradiction.

So  $\epsilon, \zeta$  must be combined with different angles from  $\beta, \gamma, \delta$ . By Lemma 16 and up to symmetry, there are three possible combinations.

In the first combination,  $\beta\gamma\delta, \beta^2\epsilon, \gamma^2\zeta$  are vertices. We get  $\beta + \gamma + \delta = 2\beta + \epsilon = 2\gamma + \zeta = 3\alpha$ . Together with  $\epsilon + \zeta = 2\alpha$ , we get  $\delta = \alpha$ , a contradiction.

In the second combination,  $\beta\gamma\delta, \beta\epsilon^2, \gamma\zeta^2$  are vertices. Similar to the first combination, we also get the contradiction  $\delta = \alpha$ .

In the third combination,  $\beta\gamma\delta, \beta^2\epsilon, \gamma\zeta^2$  are vertices. We get  $\beta + \gamma + \delta = 2\beta + \epsilon = \gamma + 2\zeta = 3\alpha$ . Together with  $\epsilon + \zeta = 2\alpha$ , we do not get a contradiction as before. However, if  $\beta\gamma\delta, \beta^2\epsilon, \gamma\zeta^2$  are the only vertices, then the total

number of  $\beta$  will be strictly bigger than the total number of  $\delta$ . Since both total numbers are actually 12, we conclude that these cannot be all the vertices. By Lemma 16, the only other possible vertices are  $\delta\epsilon^2$ ,  $\delta^2\zeta$ . If  $\delta\epsilon^2$  is a vertex, then  $\delta + 2\epsilon = 3\alpha$ . Adding this to the equations above, we get all angles equal, a contradiction. If  $\delta^2\zeta$  is a vertex, then we get a similar contradiction.

**Case 5.7.2:** There is no  $\beta\gamma\delta$ -type vertex.

We may assume  $\beta^2\gamma$  is a vertex. Since there is no  $\beta\gamma\delta$ -type vertex, by Lemma 16, the only other vertex involving  $\beta$  is  $\beta\theta^2$  for a unique  $\theta$ , and the only other vertex involving  $\gamma$  is  $\gamma^2\rho$  for a unique  $\rho$ . Since each of three distinct angles  $\delta, \epsilon, \zeta$  must appear at some vertex, there must be some pairing among  $\delta, \epsilon, \zeta$ . Without loss of generality, therefore, we may assume both  $\beta^2\gamma$ ,  $\delta^2\epsilon$  are vertices. Moreover, by symmetry, we may further assume that  $\zeta$  is paired with  $\beta$  or  $\gamma$ . Given the existence of  $\beta^2\gamma$ , the only possibilities are  $\beta\zeta^2$  and  $\gamma^2\zeta$ . So either  $\beta^2\gamma$ ,  $\delta^2\epsilon$ ,  $\beta\zeta^2$  are vertices, or  $\beta^2\gamma$ ,  $\delta^2\epsilon$ ,  $\gamma^2\zeta$  are vertices.

If  $\beta^2\gamma$ ,  $\delta^2\epsilon$ ,  $\beta\zeta^2$  are vertices, then  $2\beta + \gamma = 2\delta + \epsilon = \beta + 2\zeta = 3\alpha$ . Combined with  $\beta + \gamma + \delta + \epsilon + \zeta = 5\alpha$  from Lemma 14, we get the following pairwise relations

$$\begin{aligned} 3\beta + 2\delta &= 5\alpha, & 3\beta - \epsilon &= 2\alpha, & 3\gamma - 4\delta &= -\alpha, & 3\gamma + 2\epsilon &= 5\alpha, \\ \gamma - 4\zeta &= -3\alpha, & -\delta + 3\zeta &= 2\alpha, & \epsilon + 6\zeta &= 7\alpha. \end{aligned}$$

The relations exclude all other vertices, which must be of  $\beta^2\gamma$ -type. Therefore the anglewise vertex combination is  $\{m\beta^2\gamma, n\delta^2\epsilon, k\beta\zeta^2\}$ . This implies that the total number of  $\beta$  is strictly bigger than the total number of  $\gamma$ . Since both numbers are 12, we get a contradiction.

If  $\beta^2\gamma$ ,  $\delta^2\epsilon$ ,  $\gamma^2\zeta$  are vertices, then  $2\beta + \gamma = 2\delta + \epsilon = 2\gamma + \zeta = 3\alpha$ . Similar to the previous case, the equations together with  $\beta + \gamma + \delta + \epsilon + \zeta = 5\alpha$  from Lemma 14 exclude all other vertices. Therefore the anglewise vertex combination is  $\{m\beta^2\gamma, n\delta^2\epsilon, k\gamma^2\zeta\}$ . Then the total number of  $\delta$  is more than the total number of  $\epsilon$ , a contradiction.  $\square$

## 6 Classification of Angle Congruent Tiling

Proposition 17 classifies the numerics of spherical tilings by 12 angle congruent pentagons. In this section, we study how the numerical information can fit the combinatorial structure in Figure 8, similar to what we have done for edges in Section 4.

In Section 4, we introduced the notations  $P_i$ ,  $E_{ij}$ ,  $V_{ijk}$  for the tiles, the edges and the vertices in Figure 8. In this section, we denote by  $A_{i,jk}$  the angle of the tile  $P_i$  at the vertex  $V_{ijk}$ . We will even write  $V_{ijk} = \alpha\beta\gamma$  if we know  $V_{ijk}$  is an  $\alpha\beta\gamma$ -vertex.

Inside a specified tile, we can also name a vertex by its angle and an edge by the two angles at the ends. For example, in Figure 2,  $A_{1,23}$  is a  $\delta$ -angle,  $A_{1,26}$  and  $A_{1,34}$  are  $\alpha$ -angles. Moreover,  $E_{12}$  and  $E_{13}$  are  $\alpha\delta$ -edges of  $P_1$ .

In this section, we let  $\alpha = \frac{2}{3}\pi$  and assume  $\alpha, \beta, \gamma, \dots$  are always distinct.

There is not much to classify for the angle combination  $\alpha^5$ . Basically, we assign  $\alpha$  to all angles in Figure 8. This can be realized by the regular dodecahedron tiling.

There are many tilings for the angle combination  $\alpha^3\beta\gamma$ . Fortunately, the classification is not needed in the proof of the main theorem. We simply present some examples in Figure 17. Note that for any thick line in any tiling, we can exchange the angles  $\beta$  and  $\gamma$  along the line and still get a tiling.

**Proposition 18.** *The spherical tilings by 12 angle congruent pentagons with angles  $\alpha, \alpha, \beta, \beta, \gamma$ , where  $\alpha = \frac{2\pi}{3}$  and  $\alpha, \beta, \gamma$  are distinct, are given by Figure 20 up to symmetry. Note that in the tiling on the right, we can exchange the four angles  $\beta$  and  $\gamma$  around any thick line and still get a tiling.*

As an example of the exchange, on the right of Figure 20, we can exchange  $\beta$  and  $\gamma$  around the edge  $E_{15}$  to get  $A_{1,45} = A_{5,16} = \gamma$  and  $A_{1,56} = A_{5,14} = \beta$ . But we should keep  $A_{4,15} = A_{6,15} = \beta$ . The result is also an angle congruent tiling.

*Proof.* We are in the third case of Proposition 17. The anglewise vertex combination is  $\{8\alpha^3, 12\beta^2\gamma\}$ , which implies the following AVC condition.

**AVC**(for  $\alpha^2\beta^2\gamma$ ): Any vertex is either  $\alpha^3$  or  $\beta^2\gamma$ .

Up to symmetry, there are four possible ways of arranging the angles in the pentagon, described in Figure 18. We start by assuming the angles of the tile  $P_1$  in Figure 8 are arranged as in Figure 18.

### Arrangement 1

The case is the left of Figure 19. By  $A_{1,26} = \gamma$  and AVC, we get  $V_{126} = \beta^2\gamma$ , so that  $A_{2,16} = A_{6,12} = \beta$ . By  $A_{1,23} = A_{1,34} = \alpha$  and AVC, we get  $V_{123} = V_{134} = \alpha^3$ , so that  $A_{2,13} = A_{3,12} = A_{3,14} = A_{4,13} = \alpha$ . By  $A_{2,13} = \alpha$ ,

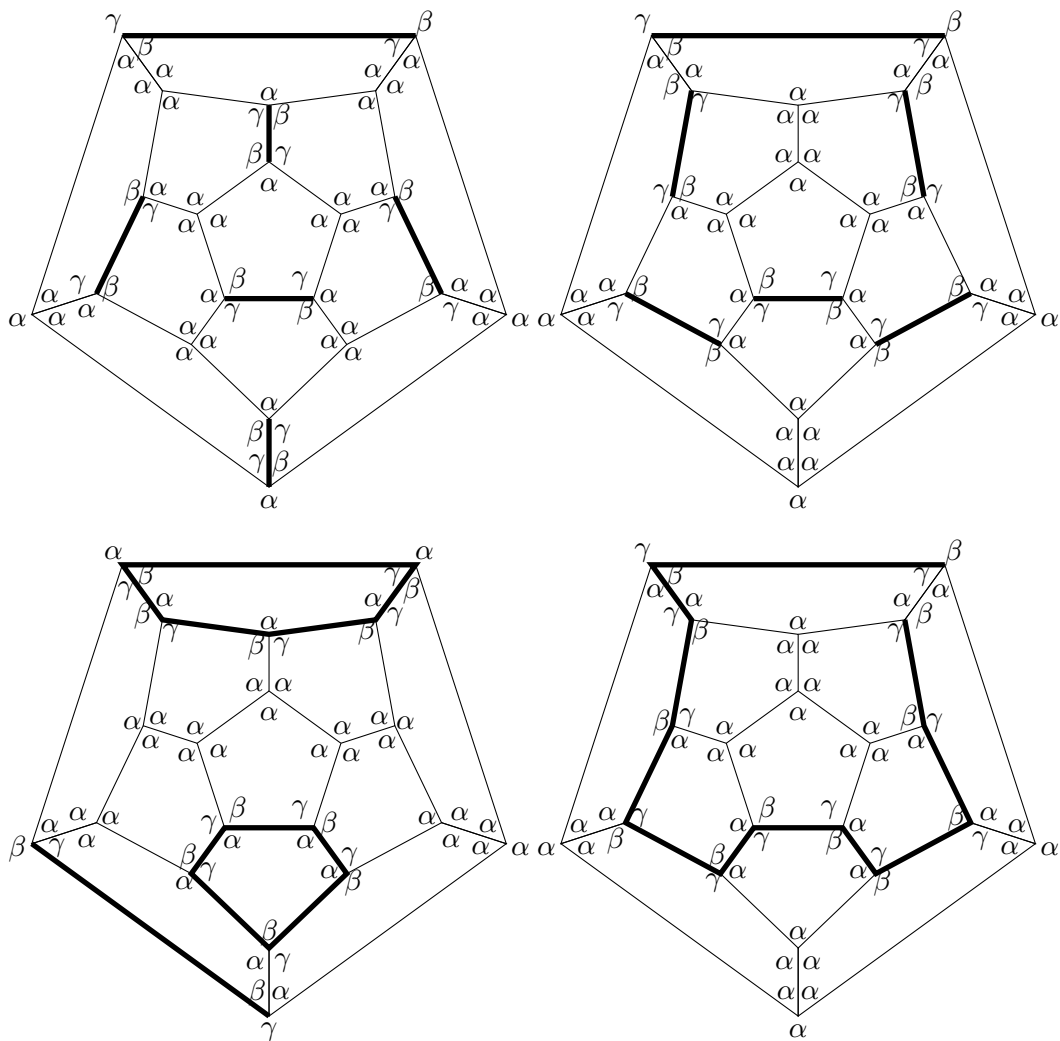


Figure 17: Some tilings for the angle combination  $\alpha^3\beta\gamma$ .

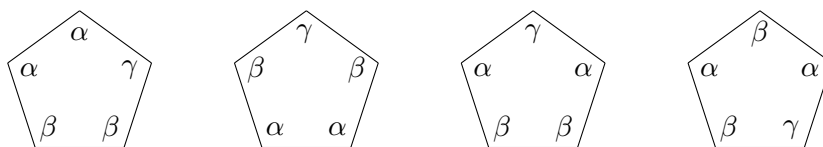


Figure 18: Four possible arrangements for the angle combination  $\alpha^2\beta^2\gamma$ .

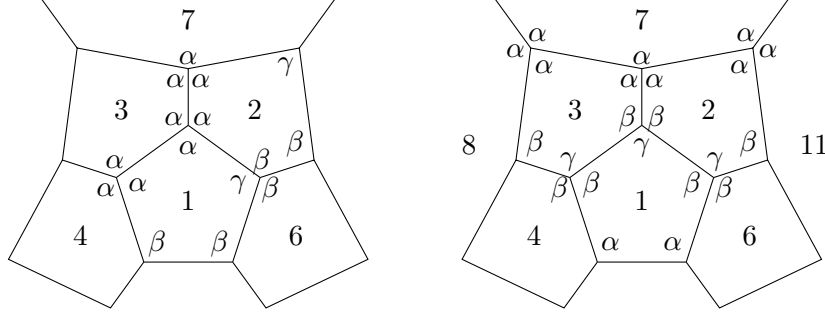


Figure 19: Tilings for the angle combination  $\alpha^2\beta^2\gamma$ .

$A_{2,16} = \beta$ , we get all the angles of  $P_2$ . Then by  $A_{2,37} = \alpha$  and AVC, we get  $V_{237} = \alpha^3$ , so that there are three  $\alpha$  in  $P_3$ , a contradiction.

### Arrangement 2

The case is the right of Figure 19. By  $A_{1,23} = \gamma$  and AVC, we get  $A_{2,13} = A_{3,12} = \beta$ . By  $A_{1,26} = \beta$  and AVC, we get  $V_{126} = \beta^2\gamma$ , so that  $A_{2,16} = \beta$  or  $\gamma$ . Since the two  $\beta$  in  $P_2$  are not adjacent, we get  $A_{2,16} = \gamma$ . Together with  $A_{2,13} = \beta$ , we get all the angles of  $P_2$ . By the same reason, we get all the angles of  $P_3$ . Then by AVC, we get  $V_{237} = V_{378} = V_{27\overline{11}} = \alpha^3$ , so that there are three  $\alpha$  in  $P_7$ , a contradiction.

### Arrangement 3

The case is the left of Figure 20. By AVC, we get all the angles at  $V_{123}$ ,  $V_{134}$ ,  $V_{126}$ . By  $A_{2,16} = A_{3,14} = \alpha$ ,  $A_{2,13} = A_{3,12} = \beta$ , we get all the angles of  $P_2$ ,  $P_3$ . By AVC, we further get all the angles at all the vertices of  $P_2$ ,  $P_3$ . The angles at these vertices then further determine all the angles of  $P_4$ ,  $P_6$ ,  $P_7$ ,  $P_8$ ,  $P_{\overline{11}}$ . By AVC, we again get all the angles at all the vertices of these tiles. Then we further get all the angles of  $P_5$ ,  $P_9$ ,  $P_{10}$ . Finally we get all the angles of  $P_{12}$  by AVC.

### Arrangement 4

The case is the right of Figure 20. By AVC, we get all the angles at  $V_{134}$ ,  $V_{126}$ ,  $V_{156}$ . We also find  $V_{145} = \beta^2\gamma$ , so that  $A_{5,14} = \beta$  or  $\gamma$ . Since  $A_{5,16} = \beta$  and the two  $\beta$  in  $P_5$  are not adjacent, we get  $A_{5,14} = \gamma$  and then all the angles of  $P_5$ . By AVC, we further get all the angles at  $V_{145}$ ,  $V_{459}$ ,  $V_{56\overline{10}}$ . We note that we may exchange the four angles  $\beta, \gamma$  around  $E_{15}$  without affecting the subsequent discussion.

Now we experiment with the angle  $A_{2,13}$ . By AVC, it must be either  $\beta$



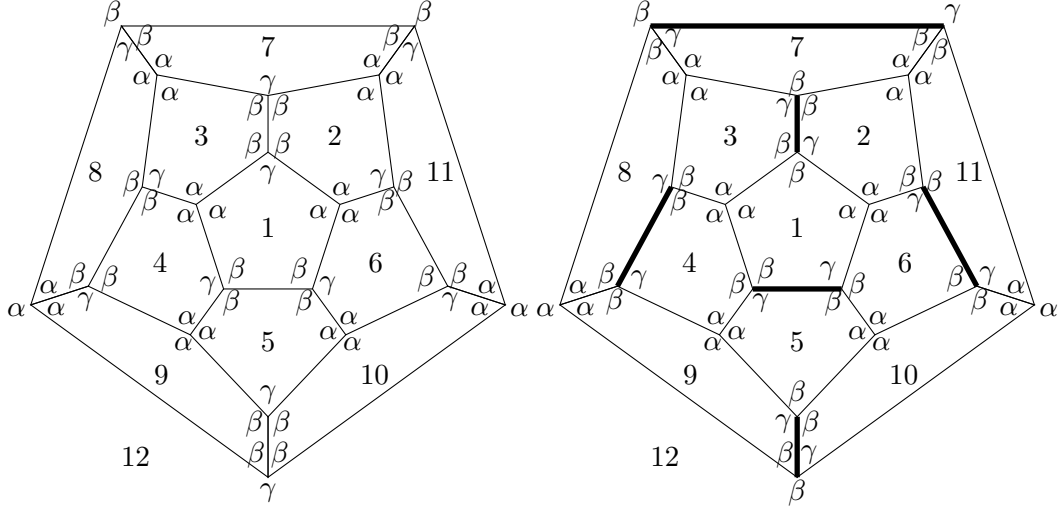


Figure 20: Tilings for the angle combination  $\alpha^2\beta^2\gamma$ , continued.

or  $\gamma$ . If  $A_{2,13} = \gamma$ , then  $A_{3,12} = \beta$  and we get all the angles of  $P_2$ . By AVC, we get  $V_{237} = \beta^2\gamma$ . Since  $A_{3,12} = \beta$  and the two  $\beta$  in  $P_3$  are not adjacent, we get  $A_{3,27} = \gamma$  and then all the angles of  $P_3$ . By AVC, we further get all the angles at  $V_{237}$ ,  $V_{2711}$ ,  $V_{378}$ . If  $A_{2,13} = \beta$ , then  $A_{3,12} = \gamma$  and we may carry out the similar argument (first on  $P_3$  and then on  $P_2$ ). The result is the same picture with the four angles  $\beta, \gamma$  around  $E_{23}$  exchanged.

We may successively carry out the same experiment around each of  $E_{48}$ ,  $E_{611}$ ,  $E_{910}$ ,  $E_{712}$ . We find that we can make similar independent choices of  $\beta, \gamma$ , and each choice determines all the angles of the pair of tiles on the two sides of the edge.  $\square$

**Proposition 19.** *The spherical tiling by 12 angle congruent pentagons with angles  $\alpha, \alpha, \beta, \gamma, \delta$ , where  $\alpha = \frac{2\pi}{3}$  and  $\alpha, \beta, \gamma, \delta$  are distinct, is given by the right of Figure 21 up to symmetry. Note that in the tiling on the right, we can exchange the four angles  $\beta$  and  $\gamma$  around any thick line and still get a tiling.*

*Proof.* We are in the fourth case of Proposition 17. The anglewise vertex combination is  $\{8\alpha^3, 12\beta\gamma\delta\}$ , which implies the following AVC condition.

**AVC**(for  $\alpha^2\beta\gamma\delta$ ): Any vertex is either  $\alpha^3$  or  $\beta\gamma\delta$ .

Up to symmetry, there are two possible ways of arranging the angles in the pentagon.

### Arrangement 1

In this arrangement, the two  $\alpha$  are adjacent. See the tile  $P_1$  on the left of Figure 21. By AVC, we get  $V_{123} = V_{134} = \alpha^3$  and  $V_{126} = \beta\gamma\delta$ . Thus  $A_{2,16} = \beta$  or  $\gamma$ , and is adjacent to  $\alpha$  in  $P_2$ . Therefore  $A_{2,16} = \beta$ . Together with  $A_{2,13} = \alpha$ , we get all the angles the of  $P_2$ . Then  $A_{2,37} = \alpha$  implies  $V_{237} = \alpha^3$ , and there are three  $\alpha$  in  $P_3$ , a contradiction.

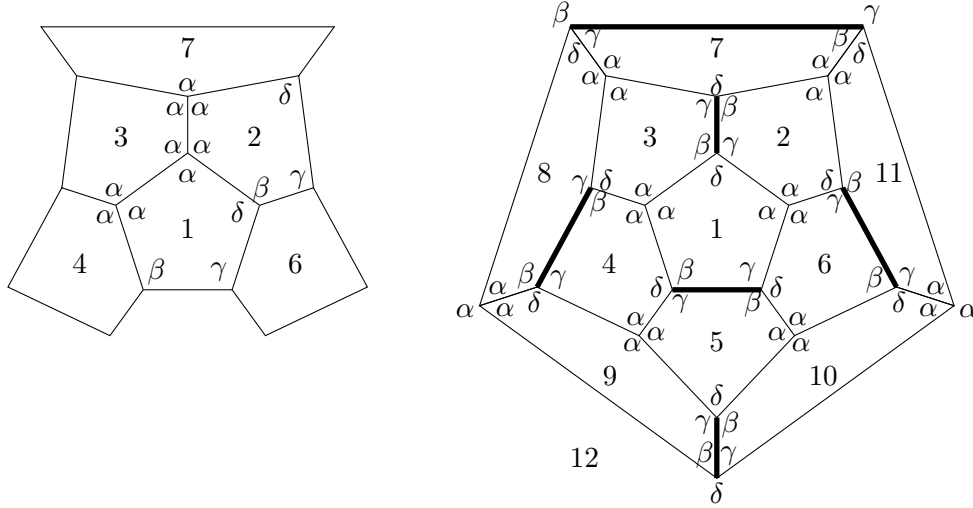


Figure 21: Tilings for the angle combination  $\alpha^2\beta\gamma\delta$ .

### Arrangement 2

In this arrangement, the two  $\alpha$  are not adjacent. See the tile  $P_1$  on the right of Figure 21. By AVC, we get  $V_{126} = V_{134} = \alpha^3$  and  $V_{123} = \beta\gamma\delta$ . If  $A_{2,13} = \gamma$ ,  $A_{3,12} = \beta$ , then we get all the angles of  $P_2$ ,  $P_3$  and all the angles at  $V_{237}$ ,  $V_{27\overline{11}}$ ,  $V_{378}$ . If  $A_{2,13} = \beta$ ,  $A_{3,12} = \gamma$ , then we get the similar result, except that the four angles  $\beta, \gamma$  around  $E_{23}$  are exchanged. Such exchange does not affect the subsequent discussion.

By  $A_{3,48} = \delta$ , we know  $A_{4,38}$ ,  $A_{8,34}$  are  $\beta, \gamma$ . We may make either choice for  $A_{4,38}$ ,  $A_{8,34}$  and then get all the angles of  $P_4$ ,  $P_8$ . Similar discussion can then be successively carried out around  $E_{6\overline{11}}$ ,  $E_{9\overline{10}}$ ,  $E_{7\overline{12}}$  and determines all the angles.  $\square$

**Proposition 20.** *If the angles in a spherical tiling by 12 angle congruent pentagons are  $\alpha, \beta, \gamma, \delta, \epsilon$ , with  $\alpha = \frac{2\pi}{3}$  and all angles distinct, then up to symme-*

try, the tiling must have anglewise vertex combination  $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$  and is given by one of the tilings on the left of Figures 24, 26, 31, 33.

*Proof.* We are in the last case of Proposition 17. The following AVC condition applies to all three possible anglewise vertex combinations in that case.

**AVC**(for  $\alpha\beta\gamma\delta\epsilon$ ): Any vertex is one of  $\alpha^3$ ,  $\alpha\beta\delta$ ,  $\alpha\gamma\epsilon$ ,  $\beta^2\gamma$ ,  $\delta^2\epsilon$ .

A consequence we will often use is that  $\beta\epsilon$ ,  $\gamma\delta$ ,  $\gamma^2$ ,  $\epsilon^2$  are forbidden at any vertex.

All three anglewise vertex combinations are symmetric with respect to the simultaneous exchange of  $\beta$  with  $\delta$  and  $\gamma$  with  $\epsilon$ . Modulo such symmetry, there are eight possible arrangements of the angles in the pentagon, described in Figure 22. We discuss each case by assuming the angles of  $P_1$  in Figure 8 are arranged as in Figure 22. Since the vertex  $\alpha\gamma\epsilon$  appears in all three anglewise vertex combinations, we will also assume  $V_{123} = \alpha\gamma\epsilon$ . This leads to the possibilities  $A_{2,13} = \gamma$ ,  $A_{3,12} = \epsilon$  and  $A_{2,13} = \epsilon$ ,  $A_{3,12} = \gamma$ .

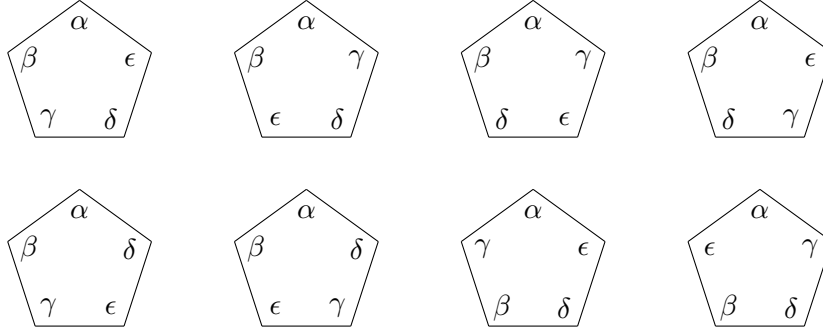


Figure 22: Eight possible arrangements for the angle combination  $\alpha\beta\gamma\delta\epsilon$ .

Since all angles are distinct, there is no ambiguity for us to call the orientation of angles given by Figure 22 *counterclockwise*, and call the other orientation *clockwise*.

**Arrangement 1, Case 1:**  $A_{2,13} = \gamma$ ,  $A_{3,12} = \epsilon$ .

The case is the left of Figure 23. Since  $A_{2,16}$  is adjacent to  $A_{2,13} = \gamma$  in  $P_2$ , we get  $A_{2,16} = \beta$  or  $\delta$ . By AVC, we get  $A_{2,16} = \delta$  and all the angles of  $P_2$ .

By  $A_{1,26} = \epsilon$ ,  $A_{2,16} = \delta$  and AVC, we get  $A_{6,12} = \delta$ . Since  $A_{6,15}$  is adjacent to  $\delta$  in  $P_6$ , we get  $A_{6,15} = \gamma$  or  $\epsilon$ . By AVC, we get  $A_{6,15} = \epsilon$  and all the angles of  $P_6$ .

By  $A_{1,56} = \delta$ ,  $A_{6,15} = \epsilon$  and AVC, we get  $A_{5,16} = \delta$ . Since  $A_{5,14}$  is adjacent to  $\delta$  in  $P_5$ , we get  $A_{5,14} = \gamma$  or  $\epsilon$ . By AVC, we get  $A_{5,14} = \epsilon$  and all the angles of  $P_5$ .

By  $A_{1,45} = \gamma$ ,  $A_{5,14} = \epsilon$  and AVC, we get  $A_{4,15} = \alpha$ . Since  $A_{4,13}$  is adjacent to  $\alpha$  in  $P_4$ , we get  $A_{4,13} = \beta$  or  $\epsilon$ . By AVC, we get  $A_{4,13} = \beta$  and all the angles of  $P_4$ .

By AVC, we further get  $A_{3,14} = \gamma$ , which is adjacent to  $A_{3,12} = \epsilon$  in  $P_3$ . This is a contradiction.

We remark that the conclusion of the case is the following: If the angles in a tile (of first arrangement) are counterclockwise oriented, and the vertex at  $\alpha$  is  $\alpha\gamma\epsilon$ , then  $\alpha, \gamma, \epsilon$  must be clockwise oriented at the vertex. By symmetry, we also know that if the angles in a tile are clockwise oriented, then  $\alpha, \gamma, \epsilon$  must be counterclockwise oriented at the vertex.

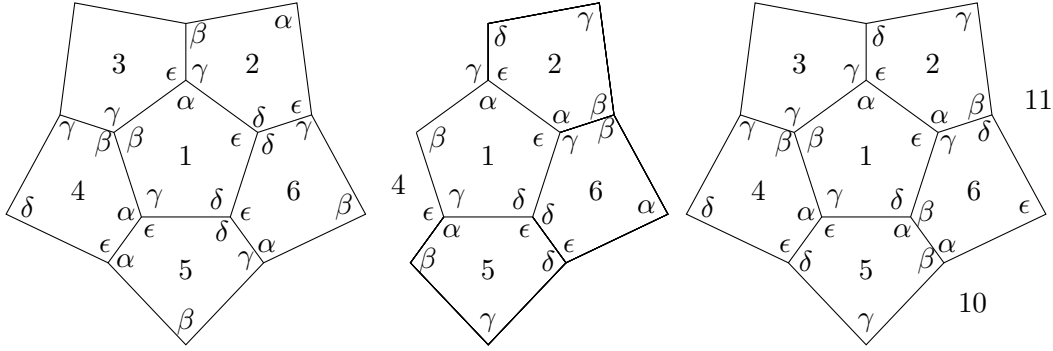


Figure 23: Tilings for the first arrangement.

**Arrangement 1, Case 2:**  $A_{2,13} = \epsilon$ ,  $A_{3,12} = \gamma$ .

The case is the middle and the right of Figure 23. Since  $A_{2,16}$  is adjacent to  $\epsilon$  in  $P_2$ , we get  $A_{2,16} = \alpha$  or  $\delta$ .

If  $A_{2,16} = \delta$ , then we get all the angles of  $P_2$  and  $A_{2,6\overline{11}} = \gamma$  in particular. By AVC, we also get  $A_{6,12} = \delta$ . Being adjacent to  $A_{6,12} = \delta$  in  $P_6$ , one of  $A_{6,15}$ ,  $A_{6,2\overline{11}}$  is  $\gamma$ . This contradicts to AVC at  $V_{156}$  or  $V_{26\overline{11}}$ .

Therefore we must have  $A_{2,16} = \alpha$ . Then we get all the angles of  $P_2$  and  $A_{6,12} = \gamma$ . Now consider the two possible orientations of  $P_6$ .

In the middle of Figure 23,  $P_6$  is counterclockwise oriented. By AVC, we get  $A_{5,16} = \epsilon$ , and the angle  $A_{5,14}$  adjacent to  $\epsilon$  in  $P_5$  must be  $\alpha$ . This determines all the angles of  $P_5$ . Now  $P_5$  is counterclockwise oriented, and  $\alpha, \gamma, \epsilon$

is also counterclockwise oriented at  $V_{145}$ . This contradicts to the conclusion of Case 1.

On the right of Figure 23,  $P_6$  is clockwise oriented. By AVC, we get  $A_{5,16} = \alpha$ . Then the angle  $A_{5,6\overline{10}}$  adjacent to  $\alpha$  in  $P_5$  is  $\beta$  or  $\epsilon$ . If  $A_{5,6\overline{10}} = \epsilon$ , then  $P_6$  is clockwise oriented, and  $\alpha, \gamma, \epsilon$  are also clockwise oriented at  $V_{56\overline{10}}$ . This contradicts to the conclusion of Case 1. Therefore we must have  $A_{5,6\overline{10}} = \beta$ . Then we get all the angles of  $P_5$  and by AVC, we further get  $A_{4,15} = \alpha$ . The clockwise orientation of  $\alpha, \gamma, \epsilon$  at  $V_{145}$  and the conclusion of Case 1 imply that  $P_4$  must be counterclockwise oriented. This determines all the angles of  $P_4$ . Then by AVC, we get  $A_{3,14} = \gamma$ , and there are two  $\gamma$  in  $P_3$ , a contradiction.

**Arrangement 2, Case 1:**  $A_{2,13} = \gamma$ ,  $A_{3,12} = \epsilon$ .

Consider the two possible orientations of  $P_3$ . The counterclockwise case is the left of Figure 24. By AVC, we may successively determine all the angles of  $P_2, P_7, P_{11}, P_6, P_5, P_{10}, P_{12}, P_8, P_4, P_9$ . There is no contradiction, and we get an angle congruent tiling with anglewise vertex combination  $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$ .

On the right of Figure 24,  $P_3$  is clockwise oriented. By AVC, we get  $A_{4,13} = \alpha$ , then get all the angles of  $P_4$ , and further get  $A_{5,14} = \alpha$ . Now one of  $A_{5,16}, A_{5,49}$  adjacent to  $\alpha$  in  $P_5$  is  $\gamma$ . This contradicts to AVC at  $V_{156}$  or  $V_{459}$ .

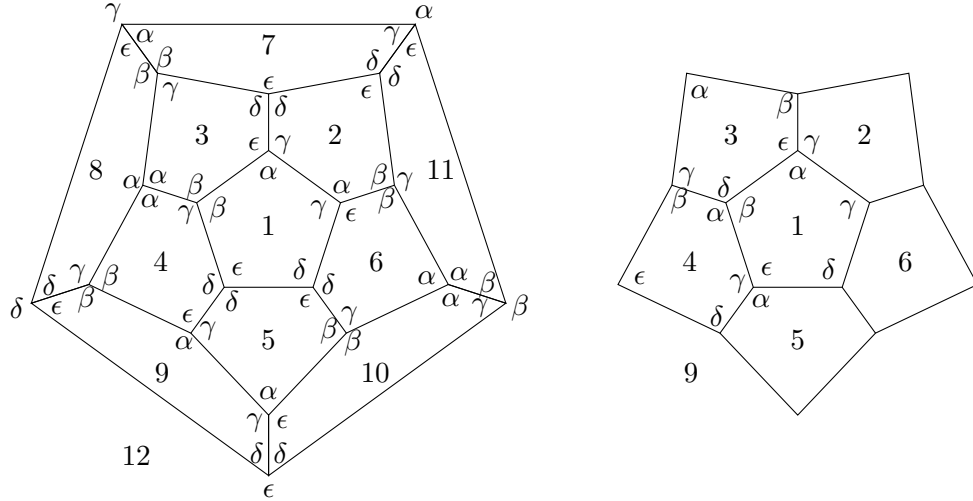


Figure 24: Tilings for the second arrangement.

**Arrangement 2, Case 2:**  $A_{2,13} = \epsilon$ ,  $A_{3,12} = \gamma$ .

By AVC, we get all the angles of  $P_2$  and  $A_{6,12} = \beta$ . If  $P_6$  is counterclockwise oriented, as on the left of Figure 25, then by AVC, we get  $A_{11,26} = \alpha$ . Now one of  $A_{11,27}$ ,  $A_{11,6\overline{10}}$  adjacent to  $\alpha$  in  $P_{11}$  is  $\gamma$ . This contradicts to AVC at  $V_{27\overline{11}}$  or  $V_{6\overline{10}\overline{11}}$ . If  $P_6$  is clockwise oriented, as on the right of Figure 25, then by AVC, we get all the angles of  $P_5, P_4$  and further get  $A_{3,14} = \alpha$ . Together with  $A_{3,12} = \gamma$ , we get  $A_{3,84} = \beta$ . This contradicts to AVC at  $V_{348}$ .

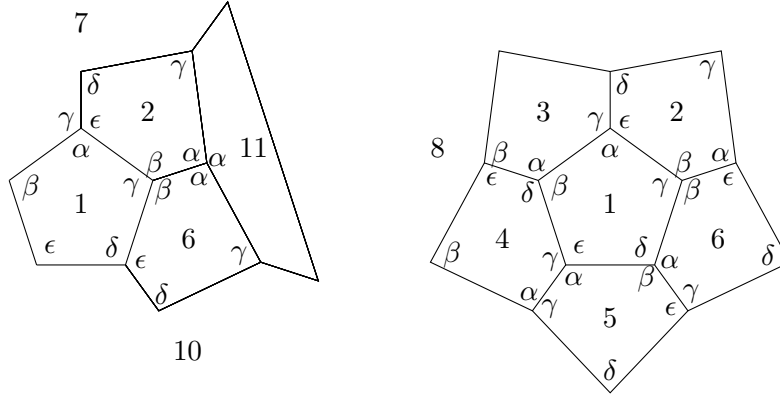


Figure 25: Tilings for the second arrangement, continued.

**Arrangement 3, Case 1:**  $A_{2,13} = \gamma$ ,  $A_{3,12} = \epsilon$ .

If  $P_2$  is clockwise oriented, as on the upper right of Figure 26, then by AVC, we may successively determine all the angles of  $P_6, P_5, P_4$  and get  $A_{3,14} = \beta$ . Thus we find  $\beta$  and  $\epsilon$  adjacent in  $P_3$ , a contradiction. Similar argument shows that  $P_3$  cannot be counterclockwise oriented. Thus we get all the angles of  $P_2, P_3$  as described on the left of Figure 26. Then by AVC, we may successively determine all the angles of  $P_7, P_{11}, P_6, P_{10}, P_5, P_4, P_8, P_9, P_{12}$ . There is no contradiction, and we get an angle congruent tiling with anglewise vertex combination  $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$ .

**Arrangement 3, Case 2:**  $A_{2,13} = \epsilon$ ,  $A_{3,12} = \gamma$ .

The case is the lower right of Figure 26. By AVC, we get all the angles of  $P_3, P_4$  and further get  $A_{5,14} = \delta$ . Then the angle  $A_{5,16}$  adjacent to  $\delta$  in  $P_5$  is either  $\beta$  or  $\epsilon$ . Either way contradicts to AVC at  $V_{156}$ .

**Arrangement 4, Case 1:**  $A_{2,13} = \gamma$ ,  $A_{3,12} = \epsilon$ .

By AVC, we get all the angles of  $P_2, P_6$  and  $A_{5,16} = \beta$ . If  $P_5$  is counterclockwise oriented, as on the left of Figure 27, then by AVC, we get all the

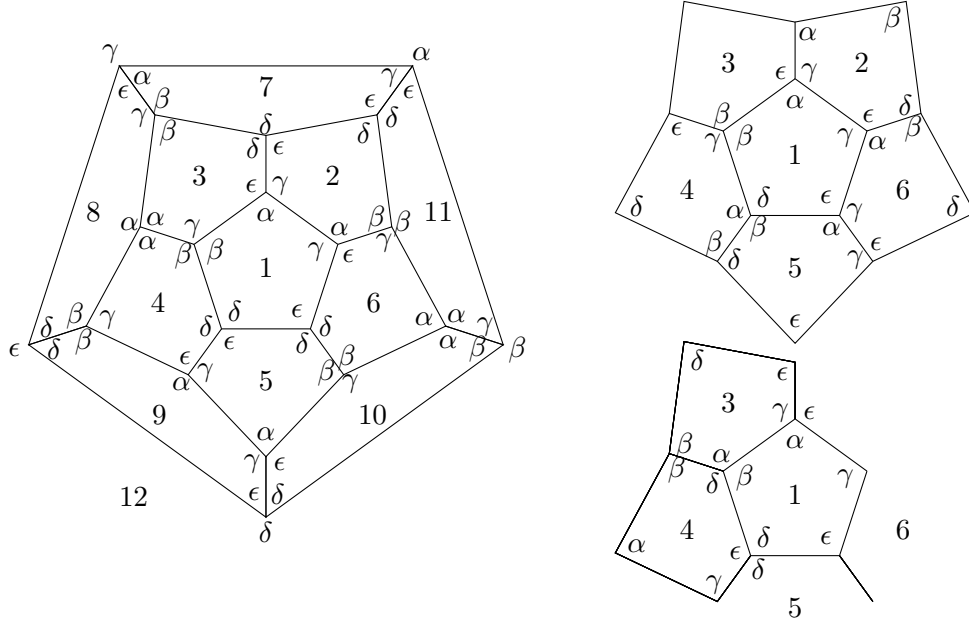


Figure 26: Tilings for the third arrangement.

angles of  $P_4$  and  $A_{3,14} = \beta$ . Then  $\beta$  and  $\epsilon$  are adjacent in  $P_3$ , a contradiction. If  $P_5$  is clockwise oriented, as on the right of Figure 27, then by AVC, we get all the angles of  $P_{10}$  and  $A_{11,6\overline{10}} = \delta$ . Then one of  $A_{11,26}$ ,  $A_{11,\overline{10}12}$  adjacent to  $A_{11,6\overline{10}}$  in  $P_{11}$  must be  $\gamma$ . This contradicts to AVC at  $V_{26\overline{11}}$  or  $A_{\overline{10}11\overline{12}}$ .

**Arrangement 4, Case 2:**  $A_{2,13} = \epsilon$ ,  $A_{3,12} = \gamma$ .

If  $P_2$  is counterclockwise oriented, as on the left of Figure 28, then by AVC, we successively get all the angles of  $P_3$ ,  $P_6$ ,  $P_{11}$  and  $A_{7,23} = \alpha$ ,  $A_{7,2\overline{11}} = \delta$ . We find  $\alpha$  and  $\delta$  adjacent in  $P_7$ , a contradiction. If  $P_2$  is clockwise oriented, as on the right of Figure 28, then by AVC, we successively get all the angles of  $P_3$ ,  $P_7$ ,  $P_8$  and  $A_{4,13} = A_{4,38} = \alpha$ . We find two  $\alpha$  in  $P_4$ , a contradiction.

**Arrangement 5, Case 2:**  $A_{2,13} = \epsilon$ ,  $A_{3,12} = \gamma$ .

We treat the second case first because the conclusion will be useful for the first case. By AVC, we get all the angles of  $P_2$ ,  $P_3$ . If  $P_4$  is clockwise oriented, as on the left of Figure 29, then by AVC, we successively get all the angles of  $P_5$ ,  $P_6$ ,  $P_9$ ,  $P_{10}$ . Then we get contradiction to AVC at  $V_{6\overline{10}11}$ .

In the argument above, we started from clockwise  $P_4$  and derived counterclockwise  $P_6$ . The same argument can be reversed, so that starting from

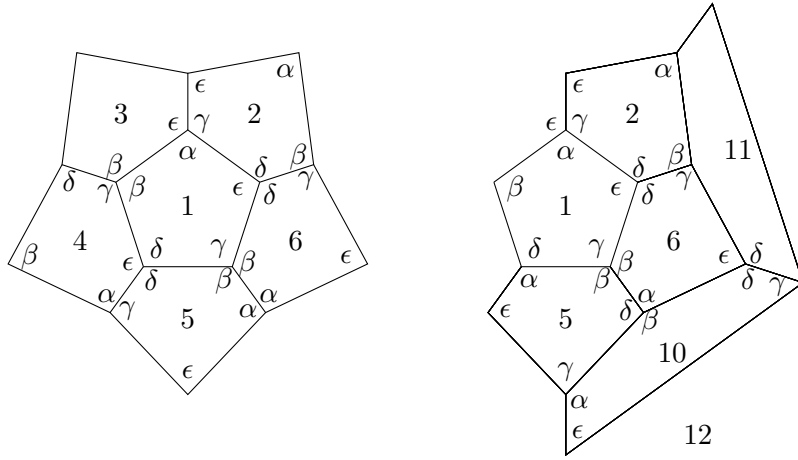


Figure 27: Tilings for the fourth arrangement.

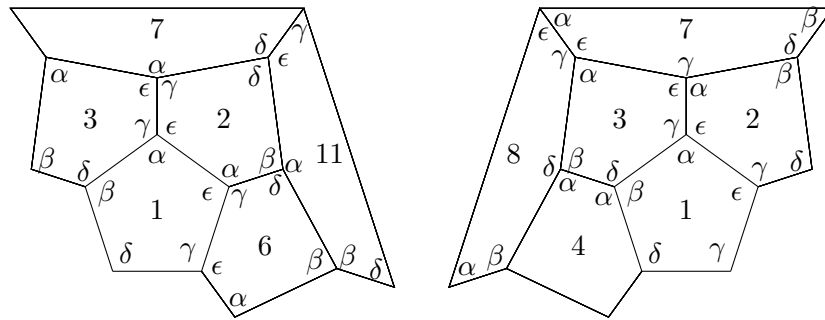


Figure 28: Tilings for the fourth arrangement, continued.



counterclockwise  $P_6$ , we can derive clockwise  $P_4$ . On the other hand, if in the left of Figure 29, we exchange  $\beta$  with  $\delta$ ,  $\epsilon$  with  $\gamma$ , and then flip the picture horizontally, we see that the argument also tells us that  $P_6$  cannot be clockwise. We conclude that  $P_6$  can be neither counterclockwise nor clockwise, a contradiction.

We remark that the conclusion of the case is the following: If the angles in a tile (of fifth arrangement) are counterclockwise oriented, and the vertex at  $\alpha$  is  $\alpha\gamma\epsilon$ , then  $\alpha, \gamma, \epsilon$  must be counterclockwise oriented at the vertex.

**Arrangement 5, Case 1:**  $A_{2,13} = \gamma$ ,  $A_{3,12} = \epsilon$ .

If  $P_2$  is counterclockwise oriented, as on the upper right of Figure 29, then by AVC, we get all the angles of  $P_6$  and  $A_{5,16} = \gamma$ . Now  $P_6$  is counterclockwise oriented and  $\alpha, \gamma, \epsilon$  are clockwise oriented at  $V_{156}$ . This contradicts to the conclusion of Case 2. If  $P_2$  is clockwise oriented, as on the lower right of Figure 29, then by AVC, we successively get all the angles of  $P_6$ ,  $P_5$ ,  $P_4$  and  $A_{3,14} = \alpha$ . We find  $\alpha, \epsilon$  adjacent in  $P_3$ , a contradiction.

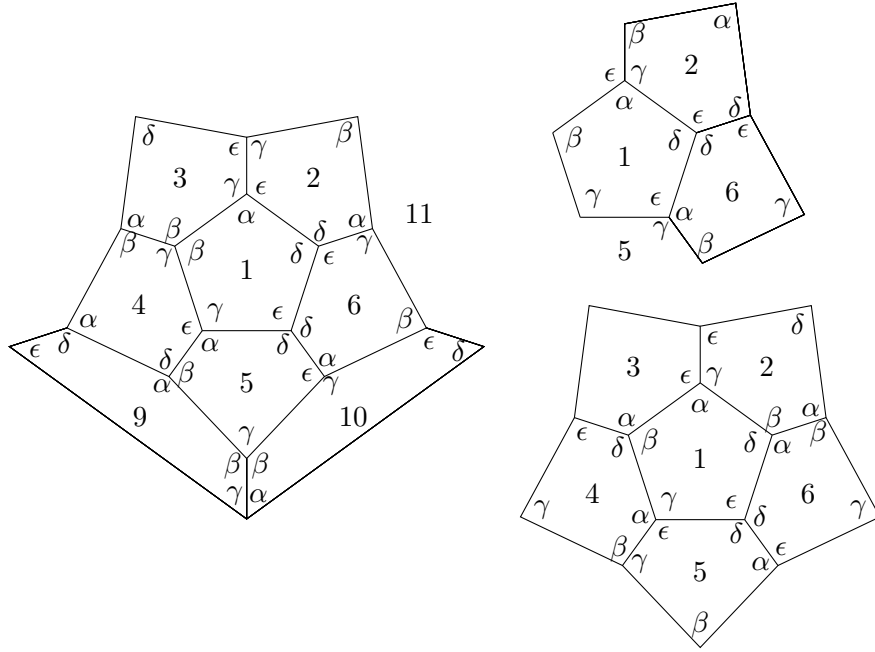


Figure 29: Tilings for the fifth arrangement.

**Arrangement 6, Case 1:**  $A_{2,13} = \gamma$ ,  $A_{3,12} = \epsilon$ .

If  $P_2$  is counterclockwise oriented, as on the left of Figure 30, then by AVC, we successively get all the angles of  $P_6$ ,  $P_5$ ,  $P_4$  and  $A_{3,14} = \alpha$ . We find  $\alpha, \epsilon$  adjacent in  $P_3$ , a contradiction. If  $P_2$  is clockwise oriented, as in the middle of Figure 30, then by AVC, we successively get all the angles of  $P_6$ ,  $P_5$ ,  $P_{10}$  and  $A_{11,26} = \beta$ ,  $A_{11,6\overline{10}} = \delta$ . We find  $\beta, \delta$  adjacent in  $P_{11}$ , a contradiction.

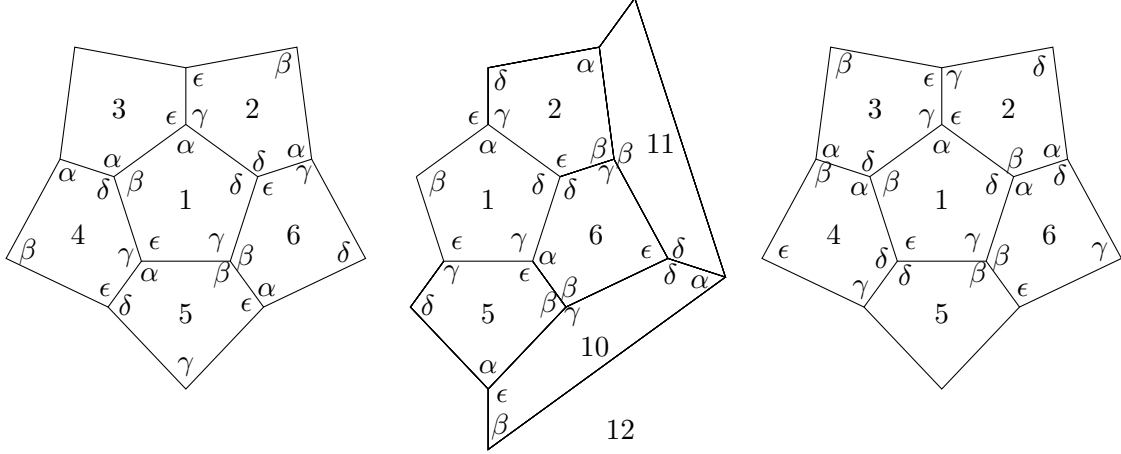


Figure 30: Tilings for the sixth arrangement.

**Arrangement 6, Case 2:**  $A_{2,13} = \epsilon$ ,  $A_{3,12} = \gamma$ .

The case is the right of Figure 30. By AVC, we successively get all the angles of  $P_2$ ,  $P_3$ ,  $P_4$ ,  $P_6$  and  $A_{5,14} = \delta$ ,  $A_{5,16} = \beta$ . We find  $\beta, \delta$  adjacent in  $P_5$ , a contradiction.

**Arrangement 7, Case 1:**  $A_{2,13} = \gamma$ ,  $A_{3,12} = \epsilon$ .

The case is the upper right of Figure 31. By AVC, we successively get all angles of  $P_2$ ,  $P_3$ ,  $P_4$ ,  $P_6$  and  $A_{5,14} = A_{5,16} = \alpha$ . We find two  $\alpha$  in  $P_5$ , a contradiction.

**Arrangement 7, Case 2:**  $A_{2,13} = \epsilon$ ,  $A_{3,12} = \gamma$ .

If  $P_2$  is clockwise oriented, as on the left of Figure 31, then by AVC, we get all the angles of  $P_6$ ,  $P_5$  and  $A_{11,26} = A_{10,56} = \gamma$ . Since  $\gamma, \epsilon$  are not adjacent, neither  $A_{10,6\overline{11}}$  nor  $A_{11,6\overline{10}}$  can be  $\epsilon$ . Then by  $A_{6,10\overline{11}} = \gamma$  and AVC, we get  $A_{10,6\overline{11}} = A_{11,6\overline{10}} = \beta$  and then all the angles of  $P_{10}$ ,  $P_{11}$ . By the angles we know so far and AVC, we successively get all the angles of  $P_9$ ,  $P_4$ ,  $P_3$ ,  $P_7$ ,  $P_8$ ,  $P_{12}$ . There is no contradiction, and we get an angle congruent tiling with anglewise vertex combination  $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$ .

By the same argument (in the left picture, exchanging  $\beta$  with  $\delta$ ,  $\epsilon$  with  $\gamma$ , and then flipping horizontally), we also conclude that, if  $P_3$  is clockwise oriented, then we get the same angle congruent tiling on the left of Figure 31. Therefore the only remaining case is that both  $P_2, P_3$  are counterclockwise oriented, as on the lower right of Figure 31. By AVC, we get all the angles of  $P_7$  and  $A_{8,37} = \delta$ . Since  $A_{8,34}$  is adjacent to  $\alpha$  in  $P_8$ , we get  $A_{8,34} = \beta$  or  $\epsilon$ . Either way contradicts to AVC at  $V_{348}$ .

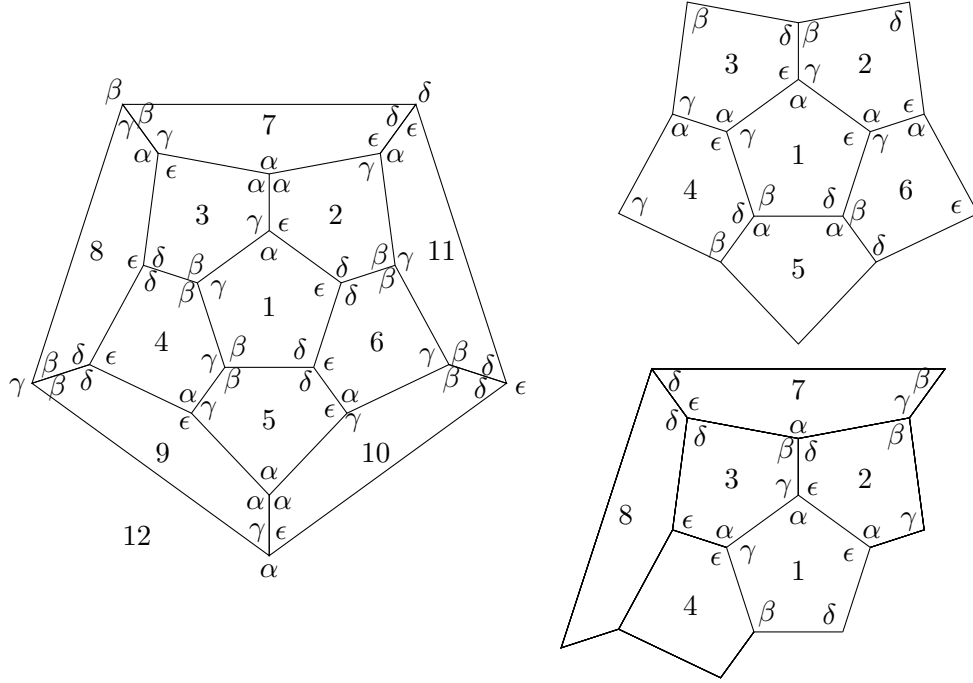


Figure 31: Tilings for the seventh arrangement.

**Arrangement 8, Case 1:**  $A_{2,13} = \gamma$ ,  $A_{3,12} = \epsilon$ .

The case is the left of Figure 32. By AVC, we successively get all the angles of  $P_2$ ,  $P_3$ ,  $P_4$ ,  $P_6$  and  $A_{5,14} = A_{5,16} = \alpha$ . We find two  $\alpha$  in  $P_5$ , a contradiction.

**Arrangement 8, Case 2:**  $A_{2,13} = \epsilon$ ,  $A_{3,12} = \gamma$ .

If  $P_2$  is counterclockwise oriented and  $P_3$  is clockwise oriented, as on the right of Figure 32, then by AVC, we successively get all the angles of  $P_7$ ,  $P_8$ ,  $P_{11}$ ,  $P_4$ ,  $P_6$  and  $A_{5,14} = \alpha$ ,  $A_{5,16} = \delta$ . We find  $\alpha, \delta$  adjacent in  $P_5$ , a contradiction. By the same argument (in the right of Figure 32, exchanging

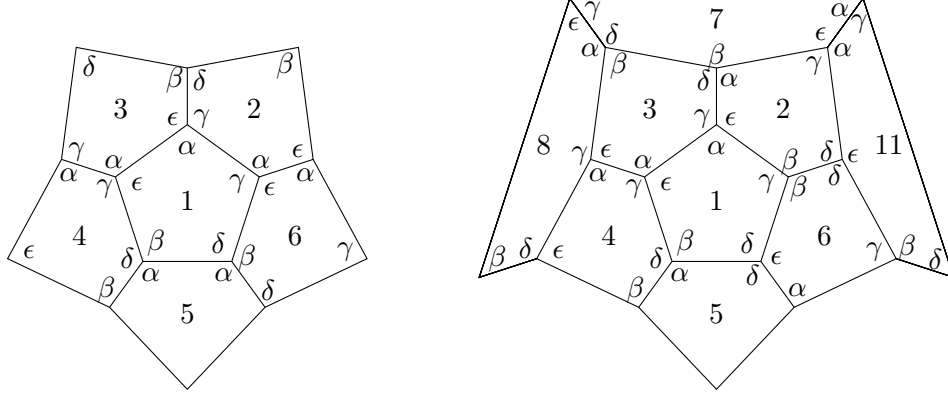


Figure 32: Tilings for the eighth arrangement.

$\beta$  with  $\delta$ ,  $\epsilon$  with  $\gamma$ , and then flipping horizontally), we also conclude that it is impossible for  $P_2$  to be clockwise oriented and  $P_3$  to be counterclockwise oriented. Therefore  $P_2$  and  $P_3$  must have the same orientation.

If both  $P_2, P_3$  are counterclockwise oriented, as on the left of Figure 33, then by AVC, we successively get all the angles of  $P_7, P_8, P_{11}, P_4, P_6, P_5, P_9, P_{10}, P_{12}$ . There is no contradiction, and we get an angle congruent tiling with anglewise vertex combination  $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$ .

If both  $P_2, P_3$  are clockwise oriented, as on the right of Figure 33, then by AVC, we successively get all the angles of  $P_7, P_8, P_{11}, P_4, P_6, P_5$  and  $A_{9,45} = \alpha, A_{9,48} = \beta$ . We find  $\alpha, \beta$  adjacent in  $P_9$ , a contradiction.  $\square$

## 7 Proof of Main Theorem

We have studied the spherical tilings by 12 congruent pentagons from the purely edge length viewpoint and the purely angle viewpoint. In this section, we combine the two together to prove the main theorem.

The following result shows that certain combinations of edge lengths and angles cannot happen for geometrical reasons.

**Lemma 21.** *Suppose in the spherical pentagon on the left of Figure 34, three of the following four equalities hold*

$$a = b, \quad c = d, \quad \beta = \gamma, \quad \delta = \epsilon,$$

*then all four equalities hold.*

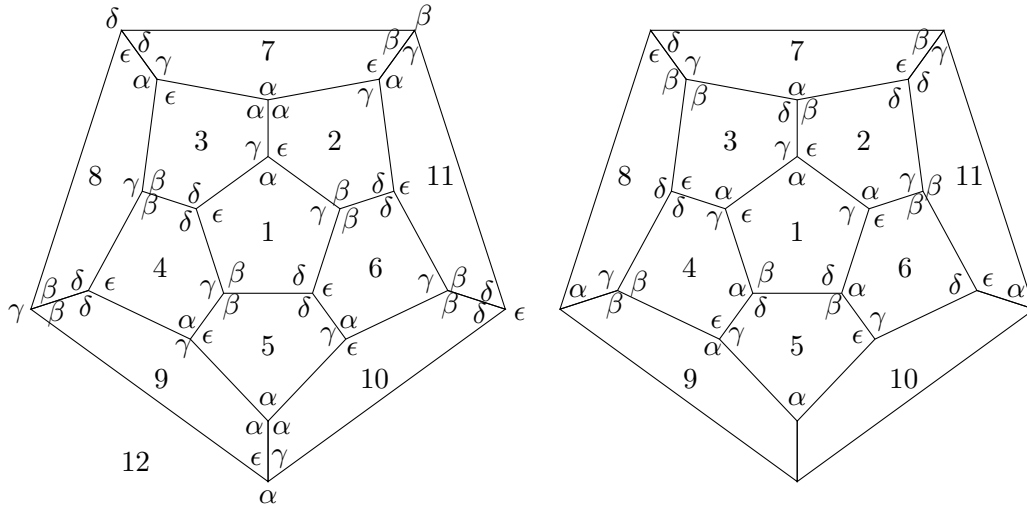


Figure 33: Tilings for the eighth arrangement, continued.

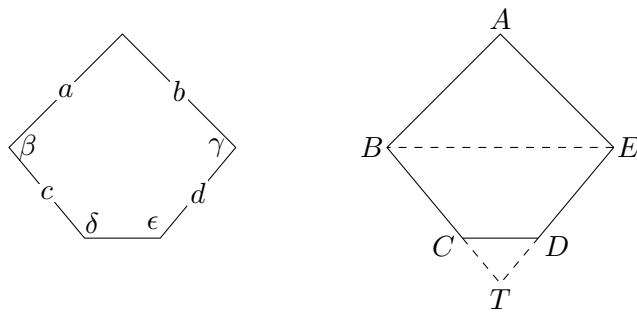


Figure 34: Geometrical constraint.

*Proof.* Add auxiliary lines as on the right of Figure 34.

Assume  $a = b$  and  $\beta = \gamma$ . Then  $a = b$  implies  $\triangle ABE$  is an isosceles triangle, and  $\beta = \gamma$  further implies that  $\triangle TBE$  is also an isosceles triangle. Then  $c = d$  if and only if  $TC$  and  $TD$  have the same length. This is further equivalent to  $\triangle TCD$  being an isosceles triangle, which is the same as  $\delta = \epsilon$ .

Assume  $c = d$  and  $\epsilon = \delta$ . Then  $\epsilon = \delta$  implies  $\triangle TCD$  is an isosceles triangle, and  $c = d$  further implies  $\triangle TBE$  is also an isosceles triangle. Then  $\beta = \gamma$  is equivalent to  $\angle ABE = \angle AEB$ , which means  $a = b$ .  $\square$

We recall the notations  $P_i, E_{ij}, V_{ijk}$  and the names  $a$ -edge,  $abc$ -vertex introduced in Section 4. We recall the names  $\alpha\beta\gamma$ -vertex,  $\alpha\beta\gamma$ -type vertex introduced in Section 5. We also recall the notation  $A_{i,jk}$  and the names  $\alpha$ -angle,  $\alpha\beta$ -edge in a tile introduced in Section 6.

We can also name an angle by the two edges around the angle. For example, the angle  $A_{1,23}$  in Figure 2 is an  $ab$ -angle, and  $A_{1,34}$  is an  $a^2$ -angle.

We use  $a, b, c, \dots$  to denote distinct edge lengths. We use  $\alpha, \beta, \gamma, \dots$  to denote distinct angles. Moreover,  $\alpha$  will always be  $\frac{2\pi}{3}$ .

*Proof of Main Theorem.* The proof is divided according to the edge length combination in the pentagon.

**Case  $a^5$ .**

If the angle combination is  $\alpha^5$ , then the pentagon is regular, and the tiling is the regular dodecahedron.

Now consider the middle three cases in Proposition 17. Up to symmetry, the angle combination  $\alpha^3\beta\gamma$  can be arranged as the first or the second in Figure 35. By Proposition 18 and up to symmetry, the angle combination  $\alpha^2\beta^2\gamma$  can be arranged as the third or the fourth in Figure 35. By Proposition 19 and up to symmetry, the angle combination  $\alpha^2\beta\gamma\delta$  can be arranged as the fifth in Figure 35.

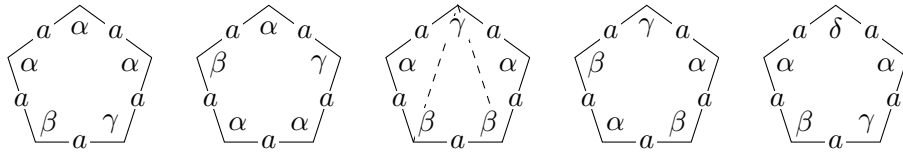


Figure 35: Combinations of  $a^5$  with  $\alpha^3\beta\gamma$ ,  $\alpha^2\beta^2\gamma$ ,  $\alpha^2\beta\gamma\delta$ .

By Lemma 21, we have  $\beta = \gamma$  in all except the third arrangement. In the third arrangement, divide the pentagon into three triangles. Note that the

three triangles (and therefore the pentagon also) are completely determined by  $a$ . Moreover, the area of the two triangles on the side are strictly increasing in  $a$ . The length of the two dividing lines are strictly increasing in  $a$ , and the area of the triangle at the middle is also strictly increasing in  $a$ . Therefore the area of the pentagon is strictly increasing in  $a$ . So there is a unique  $a$ , such that the area of the pentagon is  $\frac{1}{12}$  of the area of the sphere. Since this area is reached by the pentagon in the regular dodecahedron, we conclude that the third arrangement is the regular one.

Finally, for the angle combination  $\alpha\beta\gamma\delta\epsilon$ , we get four possible tilings from Proposition 20. Any one exists as long as there is a relevant pentagon with all sides equal. After suitable permutation among  $\beta, \gamma, \delta, \epsilon$ , these are included in the first four tilings in Figure 1.

**Case  $a^4b$ .**

Recall that the edgewise vertex combination is  $\{8a^3, 12a^2b\}$ . We will not use the complete classification in Proposition 9. Instead, we will study the edge-angle combinations directly.

**Subcase  $\alpha^5$ .**

In this case,  $b$  is completely determined by  $a$ , and the area of the pentagon is strictly increasing in  $a$ . Therefore there is only one  $a$  such that the area is  $\frac{1}{12}$  of the area of the sphere. This is the regular dodecahedron tiling. Therefore we get  $a = b$ , a contradiction.

**Subcase  $\alpha^3\beta\gamma$ .**

In addition to the edgewise vertex combination  $\{8a^3, 12a^2b\}$ , Proposition 17 also tells us the anglewise vertex combination  $\{8\alpha^3, 12\alpha\beta\gamma\}$ .

If  $\beta$  and  $\gamma$  are separated in the pentagon, as on the left of Figure 36, then up to symmetry, there are three ways of assigning  $b$ , as  $\alpha^2$ -edge or as one of two  $\alpha\beta$ -edges. Lemma 21 implies that two such assignments lead to  $\beta = \gamma$ , a contradiction. The only assignment that does not lead to a contradiction is the left of Figure 36. Similarly, if  $\beta$  and  $\gamma$  are adjacent in the pentagon, then among three possible ways (up to symmetry) of assigning  $b$ , one leads to a contradiction and we are left with the middle and the right of Figure 36.

We start the further discussion by assuming the tile  $P_1$  in Figure 8 is given by one of the pentagons in Figure 36.

The *first combination* is the left of Figure 37. Since the  $b$ -edge is also an  $\alpha\beta$ -edge of  $P_5$ , the angles  $A_{5,14}$ ,  $A_{5,16}$  must be  $\alpha, \beta$ . By AVC (anglewise vertex combination, which means that only  $\alpha^3$  and  $\alpha\beta\gamma$  are vertices in our subcase), we cannot have  $\beta^2$  at a vertex. Therefore we get  $A_{5,14} = \alpha$ ,  $A_{5,16} = \beta$ . By

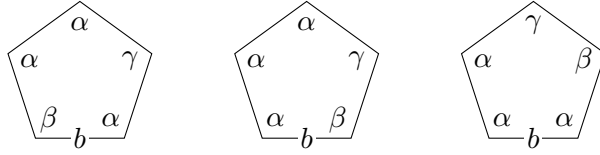


Figure 36: Combinations of  $a^4b$  and  $\alpha^3\beta\gamma$ .

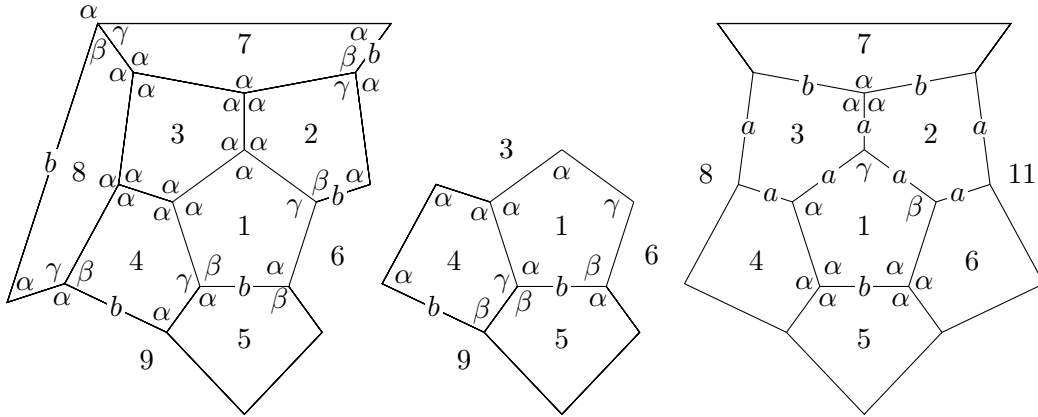


Figure 37: Tilings for the combinations of  $a^4b$  and  $\alpha^3\beta\gamma$ .



AVC, we further get  $A_{4,15} = \gamma$ , so that the  $b$ -edge of  $P_4$  is either  $E_{34}$  or  $E_{49}$ . We note that, for the current combination, the choice of the  $b$ -edge together with the location of  $\gamma$  determine all the angles of a tile. The choice  $E_{34} = b$  gives all the angles of  $P_4$  and  $A_{4,13} = \alpha$ ,  $A_{4,38} = \beta$ . By AVC, we get  $A_{3,14} = \alpha$ , and since  $b$  is an  $\alpha\beta$ -edge of  $P_3$ , we further get  $A_{3,48} = \beta$ . But  $A_{4,38} = A_{3,48} = \beta$  contradicts to AVC at  $V_{348}$ . Therefore we must have  $E_{49} = b$ . Together with  $A_{4,15} = \gamma$ , we get all the angles of  $P_4$ , as described in the picture.

So we saw  $P_1$  completely determines  $P_4$ . Note that the edge  $E_{14}$  shared by the two tiles is opposite to  $\gamma$  in  $P_1$ . Using this as a guidance,  $P_4$  determines  $P_8$ ,  $P_8$  determines  $P_7$ , and  $P_7$  determines  $P_2$ . Then by AVC, all vertices of  $P_3$  are  $\alpha^3$ -vertices, so that all the angles of  $P_3$  are  $\alpha$ , a contradiction.

The *second combination* is the middle of Figure 37. Similar to the first arrangement, by AVC and the fact that the  $b$ -edge is an  $\alpha\beta$ -edge, we get  $A_{5,16} = \alpha$ ,  $A_{5,14} = \beta$ . By AVC again, we get  $A_{4,15} = \gamma$ , and the  $b$ -edge of  $P_4$  is either  $E_{34}$  or  $E_{49}$ . If  $E_{34} = b$ , then we get all the angles of  $P_4$  and  $A_{4,13} = \beta$ . By AVC, we get  $A_{3,14} = \gamma$ , contradicting to the fact that  $b$  is an  $\alpha\beta$ -edge in  $P_3$ . Therefore we must have  $E_{49} = b$ . This determines all the angles of  $P_4$ , as described in the picture.

So we saw  $P_1$  completely determines  $P_4$ , just like the first arrangement. By the same guidance, we can successively get all the angles of  $P_8, P_7, P_2$  and find all the angles of  $P_3$  to be  $\alpha$ , a contradiction.

The *third combination* is the right of Figure 37. The combination has the special property that  $\gamma$  and  $b$  are opposite to each other in a tile and determine the location of each other. Moreover, since  $b$  is an  $\alpha^2$ -edge in  $P_5$ , we get  $A_{5,14} = A_{5,16} = \alpha$ . By AVC, this implies that both ends of any  $b$ -edge are  $\alpha^3$ -vertices. By the edgewise vertex combination, this implies that all  $\alpha\beta\gamma$ -vertices are  $\alpha^3$ -vertices. By AVC, we get  $V_{123} = V_{126} = \alpha\beta\gamma$ , so that  $V_{123} = V_{126} = \alpha^3$ . Moreover, the angles  $A_{2,13}, A_{3,12}$  are  $\alpha, \beta$ , so that the opposite edges  $E_{2\overline{11}} = E_{38} = a$ . From the four known  $a$ -edges of  $P_2$ , we get  $E_{27} = b$ . As an end of this  $b$ -edge, we get  $V_{237} = \alpha^3$ . As the edge opposite to  $A_{3,27} = \alpha$ , we get  $E_{34} = a$ . The four known  $a$ -edges of  $P_3$  imply  $E_{37} = b$ . Then we find two  $b$ -edges in  $P_7$ , a contradiction.

#### Subcase $\alpha^2\beta^2\gamma$ .

We need to assign the  $b$ -edges to the two tilings in Figure 20. The  $b$ -edge has to be a  $\rho\tau$ -edge for some (not necessarily distinct) angles  $\rho$  and  $\tau$ . Therefore the angles around the  $b$ -edge has to appear as one of the two

scenarios in Figure 38 (different  $b$ -edges may follow different scenario).



Figure 38: Angles around  $b$ -edge.

For the tiling on the left of Figure 20, the only edge in  $P_1$  fitting the scenarios in Figure 38 is the  $\beta^2$ -edge. By assigning the  $\beta^2$ -edge as  $b$  in all the tiles, we get a special case of the tiling  $T_5$  in Figure 1 and Figure 2, after suitable change of symbols.

For the tiling on the right of Figure 20, the thick edge (which is the  $\beta\gamma$ -edge) cannot be the  $b$ -edge because this would imply  $\beta = \gamma$  by Lemma 21. On the other hand, any ordinary edge is adjacent to the thick edge in some tile. Therefore in some tile, the  $b$ -edge is adjacent to the thick edge. Since the congruence between tiles preserves the thick edge (because it is the  $\beta\gamma$ -edge) and the  $b$ -edge, the  $b$ -edge is adjacent to the thick edge in every tile. Then  $E_{12}$ ,  $E_{13}$  cannot be the  $b$ -edge because they are not adjacent to the thick edge  $E_{15}$  of  $P_1$ . Moreover,  $E_{14}$ ,  $E_{16}$  cannot be the  $b$ -edge because they are not adjacent to the thick edges in  $P_4$ ,  $P_6$ . Therefore no edge of  $P_1$  can be the  $b$ -edge, a contradiction.

**Subcase  $\alpha^2\beta\gamma\delta$ .**

We need to assign the  $b$ -edges to the right of Figure 21. Again the angles around the  $b$ -edge must fit Figure 38. The only such possible edge is the  $\beta\gamma$ -edge. However,  $E_{15} = b$  implies  $\beta = \gamma$  by Lemma 21. We get a contradiction.

**Subcase  $\alpha\beta\gamma\delta\epsilon$ .**

We need to assign the  $b$ -edges to the four tilings given by Proposition 20. The angles around the  $b$ -edge must fit Figure 38.

In the tiling on the left of Figure 24, no edge of  $P_3$  fits Figure 38.

In the tilings on the left of Figures 26, 31, 33, by looking for edges of  $P_3$  in the first two cases and of  $P_1$  in the third case, we find the only edge fitting Figure 38 is the  $\beta\delta$ -edge. By assigning  $b$  to all the  $\beta\delta$ -edges and suitably changing the symbols for the angles, we get the tilings  $T_2$ ,  $T_3$ ,  $T_4$  of Figure 1.

**Case  $a^3b^2$ .**

The tiling is given on the left of Figure 14 and has the edgewise vertex combination  $\{4a^3, 4b^3, 12a^2b\}$ . The pentagon has 2  $a^2$ -angles  $\theta_1$  and  $\theta_2$ , 2  $ab$ -angles  $\rho_1$  and  $\rho_2$ , and 1  $b^2$ -angle  $\tau$ , described on the left of Figure 39. Then

$a^3$ -vertices,  $a^2b$ -vertices,  $b^3$ -vertices are respectively  $\theta_i\theta_j\theta_k$ -vertices,  $\theta_i\rho_j\rho_k$ -vertices,  $\tau^3$ -vertices. All such vertices exist.

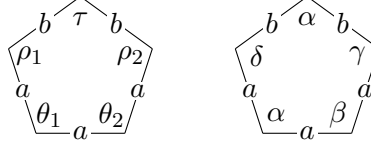


Figure 39: Pentagon for the combination of  $a^3b^2$  and  $\alpha^2\beta\gamma\delta$ .

We have  $\theta_1 + \theta_2 + \rho_1 + \rho_2 + \tau = 5\alpha$ . Since there are  $\tau^3$ -vertices, we have  $3\tau = 3\alpha$ . Therefore

$$\tau = \alpha, \quad \theta_1 + \theta_2 + \rho_1 + \rho_2 = 4\alpha.$$

By Lemma 21, we have  $\theta_1 = \theta_2$  if and only if  $\rho_1 = \rho_2$ . In this case, we have  $\theta_1 + 2\rho_1 = 3\alpha$  at a  $\theta_i\rho_j\rho_k$ -vertex. Combined with  $2\theta_1 + 2\rho_1 = 4\alpha$ , we conclude that all the angles are equal to  $\alpha$ . By Lemma 21 again, we must have  $a = b$ , a contradiction. Therefore we have  $\theta_1 \neq \theta_2$  and  $\rho_1 \neq \rho_2$ .

At a  $\theta_i\theta_j\theta_k$ -vertex, we have  $\theta_i + \theta_j + \theta_k = 3\alpha$ . Without loss of generality, we may assume the equality is either  $2\theta_1 + \theta_2 = 3\alpha$  or  $3\theta_1 = 3\alpha$ .

**Subcase**  $2\theta_1 + \theta_2 = 3\alpha$ ,  $\theta_1 \neq \theta_2$ .

Since  $\theta_1 \neq \theta_2$ , the only  $a^3$ -vertices (same as  $\theta_i\theta_j\theta_k$ -vertices) are  $\theta_1^2\theta_2$ -vertices. There are 4  $a^3$ -vertices, which involve 8 angle  $\theta_1$  and 4 angle  $\theta_2$ . Among the total of 12 angle  $\theta_1$  and 12 angle  $\theta_2$ , the remaining 4 angle  $\theta_1$  and 8 angle  $\theta_2$  must appear in the remaining 12  $a^2b$ -vertices (same as  $\theta_i\rho_j\rho_k$ -vertices).

Note that a  $\theta_1\rho_1\rho_2$ -vertex implies  $\theta_1 + \rho_1 + \rho_2 = 3\alpha$ . Combined with  $\theta_1 + \theta_2 + \rho_1 + \rho_2 = 4\alpha$  and  $2\theta_1 + \theta_2 = 3\alpha$ , we get  $\theta_1 = \alpha = \theta_2$ , a contradiction. Similarly, there is no  $\theta_2\rho_1\rho_2$ -vertex. Therefore  $a^2b$ -vertices are  $\theta_i\rho_j^2$ -vertices, and we have 4  $\theta_1\rho_j^2$ -vertices and 8  $\theta_2\rho_k^2$ -vertices. Since  $j = k$  will lead to  $\theta_1 = 3\alpha - 2\rho_j = 3\alpha - 2\rho_k = \theta_2$ , we must have  $j \neq k$ . Without loss of generality, we may assume  $j = 1$ ,  $k = 2$  and get  $\theta_1 + 2\rho_1 = 3\alpha$  and  $\theta_2 + 2\rho_2 = 3\alpha$ . Combined with  $\theta_1 + \theta_2 + \rho_1 + \rho_2 = 4\alpha$  and  $2\theta_1 + \theta_2 = 3\alpha$ , we get all the angles equal to  $\alpha$ , a contradiction.

**Subcase**  $3\theta_1 = 3\alpha$ ,  $\theta_1 \neq \theta_2$ .

We have  $\theta_1 = \alpha$  and  $\theta_2 + \rho_1 + \rho_2 = 3\alpha$ . If any of  $\theta_1\theta_2^2$ ,  $\theta_1^2\theta_2$ ,  $\theta_2^3$  is a vertex, then we will always get  $\theta_2 = \alpha = \theta_1$ , a contradiction. Therefore the

only  $a^3$ -vertices (same as  $\theta_i\theta_j\theta_k$ -vertices) are  $\theta_1^3$ -vertices. Since all 12 angle  $\theta_1$  are concentrated at the 4  $a^3$ -vertices, the 12  $a^2b$ -vertices (same as  $\theta_i\rho_j\rho_k$ -vertices) must be  $\theta_2\rho_j\rho_k$ -vertices. The possible  $\theta_2\rho_j\rho_k$ -vertices are  $\theta_2\rho_1^2$ ,  $\theta_2\rho_2^2$ ,  $\theta_2\rho_1\rho_2$ . If any two appear as vertices, then we get  $\rho_1 = \rho_2$ , a contradiction. Since both  $\rho_1$  and  $\rho_2$  must appear, we conclude that all 12  $a^2b$ -vertices are  $\theta_2\rho_1\rho_2$ -vertices.

Denote  $\theta_2 = \beta$ ,  $\rho_1 = \delta$ ,  $\rho_2 = \gamma$ . Then  $\alpha \neq \beta$  and  $\delta \neq \gamma$ , and the pentagon is given by the right of Figure 39. The discussion in the last paragraph tells us that, as far as vertices are concerned, we have

$$a^3 = \alpha^3, \quad a^2b = \beta\gamma\delta. \quad (\text{note that } \theta_2\rho_1\rho_2 = \beta\gamma\delta)$$

Now we try to fit the right of Figure 39 into the left of Figure 14, the only edge congruent tiling for  $a^3b^2$ . The two  $a^2$ -angles  $A_{1,45}$  and  $A_{1,56}$  in  $P_1$  are  $\theta_1 = \alpha$  and  $\theta_2 = \beta$ . Since  $V_{145} = a^3 = \alpha^3$  and  $V_{156} = a^2b = \beta\gamma\delta$ , we get  $A_{1,45} = \alpha$  and  $A_{1,56} = \beta$ . We also know the  $b^2$ -angle of  $P_1$  is  $A_{1,23} = \tau = \alpha$ . Knowing three angles of  $P_1$  determines the other two angles. By the same argument, we get all the angles of all tiles. After rotating the result by angle  $\frac{\pi}{5}$  in clockwise orientation and letting  $c = a$ , we get the tiling  $T_5$  in Figure 1 and Figure 2.

#### Case $a^2b^2c$ .

The tiling is given on the left of Figure 16. Since  $a, b, c$  are distinct, we can uniquely define  $\alpha_1, \alpha_2, \beta, \gamma, \delta$  as the  $a^2$ -angle, the  $b^2$ -angle, the  $ac$ -angle, the  $bc$ -angle, the  $ab$ -angle in the pentagon. After assigning these angles, we find  $3\alpha_1 = 3\alpha_2 = 2\pi$  at  $b^3$ -vertices. Therefore  $\alpha_1 = \alpha_2 = \alpha$ , and we get the tiling  $T_5$  in Figure 1 and Figure 2.

This completes the proof of the main classification theorem.  $\square$

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